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Large and Small Solutions of a Class of Quasilinear Elliptic Eigenvalue Problems

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Existence and uniqueness results for large positive solutions are obtained for a class of quasilinear elliptic eigenvalue problems in general bounded smooth domains via a generalization of a sweeping principle of Serrin. The nonlinear terms of the problems can be negative in some intervals. The existence and structure of a mountain pass solution are also discussed. We show that this solution develops to a spike layer solution. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded, connected, smooth domain, and let Δ_p be the p-Laplacian defined by $\Delta_p u = \operatorname{div}(|Du|^{p-2}\, Du)$. We consider the existence of positive solutions of the quasilinear eigenvalue problem

$$-\Delta_p u = \lambda f(u) \text{ in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

for $1 < p < \infty$, $\lambda > 0$, under appropriate smoothness conditions on f.

By a positive solution of Eq. (1.1) we mean a pair (λ, u) in $\mathbb{R}^+ \times C_0^1(\bar{\Omega})$ satisfying Eq. (1.1) in the weak sense and with $u > 0$ in Ω.

This problem appears in the study of non-Newtonian fluids. The quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant

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fluids and those with $p<2$ are called pseudoplastics. If $p=2$, they are Newtonian fluids (see, for example, [12] and its bibliography.) Other applications of such problems are found when seeking soliton-like solutions of Lorentz invariant equations; see [1, 2].

In this paper we allow f to change sign, in particular we assume f satisfies the following conditions.

(F_1) $f(0)=0$; there are precisely two numbers $0<\rho_1<\rho_2$ such that $f(\rho_1)=f(\rho_2)=0$ and $f>0$ in (ρ_1, ρ_2), $f<0$ in $(0, \rho_1)$, $\lim_{s\to 0^+} f(s)/s^{p-1} = -m<0$, $f'(s)<0$ near $s=0$, $-\infty < \lim_{s\to\rho_1^-} f(s)/(\rho_1-s)^{p-1}<0$, $0<\lim_{s\to\rho_1^+} f(s)/(s-\rho_1)^{p-1}<\infty$.

(F_2) $\int_\rho^{\rho_2} f(s)\,ds>0$ for every $\rho\in[0, \rho_2)$. We denote by $\hat{\mu}\in(\rho_1, \rho_2)$ the unique number such that $\int_0^{\hat{\mu}} f(s)\,ds=0$.

Such problems have been extensively studied by many authors; see, for example, [17–24, 27, 29, 33–35, 38, 39].

In [17], it was shown that (F_2) is a necessary condition for the existence of a positive solution u_λ of (1.1) with $\max u_\lambda\in(\rho_1, \rho_2]$. In the present paper we obtain a uniqueness result for positive solutions of (1.1) whose maximum is close to ρ_2 under the extra condition:

(F_3) There exists $\delta>0$ such that $f'(s)<0$ in $(\rho_2-\delta, \rho_2)$ and there exists $M>0$ such that $f(s)\leqslant M(\rho_2-s)^{p-1}$ for $0<s\leqslant\rho_2$.

A typical example of f satisfying (F_1)–(F_3) is

$$f(s)=\begin{cases} -ms^{p-1}\,|a-s|^{p-2}\,(a-s)\,|1-s|^{p-2}\,(1-s), & \text{for} \quad p\geqslant 2\\ -ms^{p-1}(a-s)(1-s), & \text{for} \quad 1<p<2, \end{cases}$$

where $0<a<1/4$.

Furthermore, we obtain, by the mountain pass lemma, the existence of a second positive solution $\underline{u}_\lambda$ of Eq. (1.1) and we study the structure of $\underline{u}_\lambda$.

The main results of this paper are the following theorems.

THEOREM A. *Let $f\in C^0([0,\infty))\cap C^1((0,\infty))$ satisfy (F_1)–(F_3). Then for each nonnegative function $\zeta\in C_0^\infty(\Omega)$ with $\max\zeta\in(\rho_1, \rho_2)$, there is $\lambda_0=\lambda_0(\zeta)>0$ such that for all $\lambda>\lambda_0$, (1.1) possesses exactly one solution $\bar{u}_\lambda$ satisfying $\zeta<\bar{u}_\lambda<\rho_2$ and $\lim_{\lambda\to\infty}\max\bar{u}_\lambda=\rho_2$. Moreover, for any compact set $K\subset\Omega$, $\bar{u}_\lambda\to\rho_2$ in K as $\lambda\to\infty$.*

THEOREM B. *Assume $p>2$, $\Omega\subset\mathbb{R}^N$ ($N\geqslant 3$) is a convex domain, and f satisfies the conditions (F_1)–(F_3). Then, for λ sufficiently large, there exists a* *positive solution* $\underline{u}_\lambda \not\equiv \bar{u}_\lambda$, *which is a mountain pass solution of* (1.1) *for the functional*

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |Du|^p - \lambda \int_\Omega F(u) \qquad \left(\text{where } F(u) = \int_0^u f(s)\,ds\right).$$

Moreover, $\rho_1 < \max_\Omega \underline{u}_\lambda < \rho_2$, *and* $\underline{u}_\lambda$ *has the following properties:*

(i) $C_1^* \lambda^{-N/p} \leqslant J_\lambda(\underline{u}_\lambda) \leqslant C_1 \lambda^{-N/p}$ *for* $1 < p < N$, *where* $C_1^* > 0$, $C_1 > 0$ *are independent of* λ,

(ii) $C_1^* \lambda^{-(1+p/q)} \leqslant J_\lambda(\underline{u}_\lambda) \leqslant C_1 \lambda^{-N/p}$ *for* $p \geqslant N$ *and any* $q > 0$, *where* $C_1^* > 0$, $C_1 > 0$ *are independent of* λ,

(iii) $\int_\Omega \underline{u}_\lambda^p\,dx \leqslant C_2 \lambda^{-N/p}$ *for some constant* $C_2 > 0$ *independent of* λ.

THEOREM C. *Let* $p > 2$, Ω *be as in Theorem* B, *and suppose* f *satisfies* (F_1)–(F_3) *and*

(F_4) $(s-\rho_1) f'(s) - (p-1) f(s) < 0$ *for* $s \in (\rho_1, \rho_2)$.

Suppose that $\underline{u}_\lambda$ *is the solution obtained in Theorem* B *which is such that for a* $\sigma^* > 0$ *satisfying* $\rho_1 + \sigma^* < \hat{\mu}$, *the set* $\Omega_{\lambda, \rho_1+\sigma^*} = \{x \in \Omega : \underline{u}_\lambda > \rho_1 + \sigma^*\}$ *is a connected convex set. Then, for* λ *sufficiently large,* $\underline{u}_\lambda$ *has only one local (hence global) maximum point* $P_\lambda \in \Omega$, $\operatorname{dist}(P_\lambda, \partial\Omega) \geqslant \theta > 0$; $\underline{u}_\lambda \to 0$ *outside any neighbourhood of* P_λ *and* $\underline{u}_\lambda(P_\lambda) \to w(0)$, *where* w *is the unique positive (radial) solution of*

$$\Delta_p w + f(w) = 0 \quad \text{in} \quad \mathbb{R}^N, \qquad w \to 0 \quad \text{as} \quad |x| \to \infty, \tag{1.2}$$

with $\rho_1 < w(0) < \rho_2$. *More precisely,* $\underline{u}_\lambda(\lambda^{-1/p}\cdot + P_\lambda) \to w(\cdot)$ *uniformly in* $C^1_{loc}(\Omega_\lambda)$ *where* $\Omega_\lambda = \{y\colon \lambda^{-1/p} y + P_\lambda \in \Omega\}$.

The existence and uniqueness of the positive radial solution w with $w(0) > \hat{\mu}$, $w'(0) = 0$, $w'(r) < 0$ for $r > 0$ of Eq. (1.2) under the assumptions (F_1)–(F_3) and (F_4) with $p > 2$ have been obtained in [20]. It is also shown in [20] that

$$\lim_{r\to\infty} \sup w(r)\, e^{\left(\frac{m}{p-1} - \eta\right)^{1/p} r} < \infty$$

for any $\eta \in (0, m/(p-1))$ and

$$\lim_{r\to\infty} \frac{w'(r)}{w(r)} = -\left(\frac{m}{p-1}\right)^{1/p}.$$

Such kinds of uniqueness results have also been obtained in [13] with some assumptions on f different from those in [20]. The main results in [20] are closely related to those in [8, 36], for $p = 2$.

In [20], it was proved that when $\Omega = B$, the unit ball of $\mathbb{R}^N$, f satisfies the conditions of Theorem C, (1.1) has precisely two positive solutions, and they are both radial. The following result, Theorem 2 from [4], was used in [20].

THEOREM D. *Let $p \in (1, \infty)$ and let $f = f(s)$ be continuous and bounded on $\mathbb{R}_0^+$ and satisfy:*

(a) *if $f(S) = 0$ for some $S > 0$, then there is a function $\beta \in A_p$ such that*

$$f(s) \leqslant \beta(S - s) \qquad \text{for} \quad 0 \leqslant s \leqslant S,$$

where

$$A_p = \left\{ \beta \in C(\mathbb{R}_0^+);\ \beta(0) = 0,\ \beta \text{ is nondecreasing and } \int_0^1 (s\beta(s))^{-1/p}\, ds = +\infty \right\}.$$

Assume that $u \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ satisfies

$$-\Delta_p u = f(u), \qquad u \geqslant 0 \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Suppose that $\lim_{|x| \to \infty} u(x) = 0$, $u > 0$, and there is some number $\tilde{\delta} > 0$ such that $f(s)$ is nonincreasing for $0 < s < \tilde{\delta}$ and $f(u(\cdot)) \in L^1(\mathbb{R}^N)$. Then u is radially symmetric about some $x_0 \in \mathbb{R}^N$.

Theorem D was established by using a new rearrangement technique called *continuous Steiner symmetrization* (see [3, 4]) together with the maximum principle for the p-Laplacian. Note that $\beta(s) = Ms^{p-1} \in A_p$. For other radially symmetric results similar to Theorem D for $1 < p < 2$, we refer to [11, 41].

As in [20], we shall assume that $x_0 = 0$ and $0 \in \Omega$. Otherwise, we may use a simple transformation to make this true. Theorem D and the results in [20] imply that Eq. (1.2) has precisely one positive radial solution w with $w'(r) < 0$ for $r > 0$ and $w(r) \to 0$ as $r \to \infty$.

We call $\bar{u}_\lambda$ in Theorem A a large solution of (1.1) and $\underline{u}_\lambda$ in Theorem B a small solution of (1.1).

When $p = 2$, Theorem A has been obtained by Clement and Sweers [6] by using the strong maximum principle, the strong comparison principle, and the sweeping principle of Serrin for the Laplacian Δ. When $p \neq 2$ such nice features seem to be lost or at least difficult to verify. Clement and Sweers [6] obtained uniqueness of solutions of (1.1) with $p=2$ by use of Leray–Schauder degree theory. They first showed that any solution u_λ is isolated and then calculated the index of u_λ. To prove the isolatedness of u_λ, they used linearization of the equation at u_λ. We cannot use such a method when $p \neq 2$ since the linearization of our equation seems to be very complicated. We have obtained some uniqueness results for problem (1.1) under the assumptions that f is increasing in $(0, +\infty)$ (see [23]) or Ω is a ball or an annulus (see [18, 20, 24]). The problem seems to be more difficult without such special assumptions.

It follows from a strong maximum principle in [34] and [44] that if f satisfies (F_3) and u_λ is a positive solution of (1.1), then $u_\lambda < \rho_2$ in Ω. If $f(s) \sim C(\rho_2 - s)^k$ with $0<k<p-1$ for s near ρ_2, a flat core of u_λ may occur. That is, $E = \{x \in \Omega : u_\lambda(x) = \rho_2\} \neq \emptyset$ (see [26, 29]). Notice that if $p>2$, $f \in C^0([0,\infty)) \cap C^1((0,\infty))$ satisfies (F_1)–(F_2) and $f'(\rho_2)<0$, then f does not satisfy (F_3). This case is difficult since we need information about the flat core of the large solution; we will leave the discussions to [22].

Theorems B and C have been studied in the case $p=2$ by many authors; see, for example, [10, 25, 30–32]. We overcome many technical difficulties here for $p>2$.

2. SWEEPING OUT RESULTS

In this section we give some results which are useful for the proofs of the main theorems.

LEMMA 2.1. *Assume that f satisfies* (F_1) *and* (F_2). *Let (λ, u) be a positive radial solution of the problem*

$$-\operatorname{div}(|Du|^{p-2} Du) = \lambda f(u) \quad \text{in } B, \qquad u=0 \quad \text{on } \partial B, \tag{2.1}$$

where B is the unit ball in $\mathbb{R}^N$, satisfying $\max u \leqslant \rho_2$. *Then*

(i) *if* $\max u < \rho_2$, $u'(r) < 0$ *for* $r \in (0, 1]$,

(ii) *if* $\max u = \rho_2$ *there exists* $0 \leqslant r_0 < 1$ *such that* $u \equiv \rho_2$ *in* $[0, r_0]$, $u'(r)<0$ *for* $r \in (r_0, 1]$.

Proof. Since u is a positive radial solution of Eq. (2.1), u satisfies

$$-(|u'|^{p-2} u')' - \frac{N-1}{r} |u'|^{p-2} u' = \lambda f(u). \tag{2.2}$$

Let $\tau \in [0, 1)$ be a point such that $u'(\tau) = 0$. Multiplying both sides of Eq. (2.2) by u' and integrating on $(\tau, 1)$, we obtain

$$(1 - 1/p)\, |u'(1)|^p + \int_\tau^1 \frac{N-1}{\xi}\, |u'(\xi)|^p \, d\xi + \lambda \int_{u(\tau)}^0 f(s)\, ds = 0.$$

This implies that $u(\tau) > \hat{\mu}$, the unique number such that $\int_0^{\hat{\mu}} f(s)\, ds = 0$. In particular $u(0) > \hat{\mu}$.

If $u'(r_1) = 0$ and $u'(r_2) = 0$ for $0 \leqslant r_1 < r_2 \leqslant 1$ then multiplying by u' on both sides of Eq. (2.2) and integrating on (r_1, r_2), we have

$$\int_{r_1}^{r_2} \frac{N-1}{\xi}\, |u'(\xi)|^p \, d\xi + \lambda \int_{u(r_1)}^{u(r_2)} f(s)\, ds = 0. \tag{2.3}$$

This is impossible if $u(r_2) > u(r_1)$. Hence there cannot exist a local minimum since it would have to be followed by a local maximum. Also $\max u = u(0)$ and so u is decreasing for small $r > 0$.

Since the strong maximum principle need not hold for our operator (see [27]), there may exist $0 < r_0 < 1$ such that $u(r) = u(0)$ in $[0, r_0]$. This constant value must be ρ_2. This is impossible in case (i).

In case (i) we claim that $u'(r) < 0$ for all $r \in (0, 1)$. Indeed, if not, we let r_3 be the first point with $u'(r_3) = 0$, so that u is decreasing on $(0, r_3)$ and $u(r_3) > \hat{\mu}$. Therefore $f(u(r)) > 0$ for $r \in (0, r_3)$. Writing Eq. (2.1) in the form

$$-(r^{N-1}\, |u'|^{p-2}\, u')' = \lambda r^{N-1} f(u)$$

and integrating on $(0, r_3)$ we obtain

$$\int_0^{r_3} r^{N-1} f(u(r))\, dr = 0.$$

This contradiction shows that no such r_3 exists. The rest of case (ii) is now clear. ∎

Lemma 2.2. *Assume that f satisfies* (F_1). *Then there is no positive radial solution $u(r)$ of Eq.* (2.1) *satisfying $u(1) = 0$, $u'(1) = 0$ for $u > 0$ in* $(0, 1)$, *and $u' < 0$ in $(1 - \hat{\delta}, 1)$, $\hat{\delta} > 0$.*

Proof. Suppose that there exists a solution u satisfying the listed properties. Then

$$u(r) = \lambda^{1/(p-1)} \int_r^1 \left[s^{1-N} \int_s^1 \xi^{N-1}(-f(u(\xi)))\, d\xi \right]^{1/(p-1)} ds$$

for $r \in (1-\hat{\delta}, 1)$. Since $f(u) \geqslant -(m+1)\, u^{p-1}$ for sufficiently small $u > 0$, we have

$$u(r) \leqslant \lambda^{1/(p-1)}((m+1)/N)^{1/(p-1)}\,[(1-r^N)/r^{N-1}]^{1/(p-1)}\,(1-r)\,u(r),$$

for $r \in (1-\hat{\delta}, 1)$. Thus, $u \equiv 0$ in $(1-\hat{\delta}, 1)$ for $\hat{\delta}$ sufficiently small. This is a contradiction. ∎

Let $\varepsilon > 0$ (which will be chosen to be sufficiently small below). We make an extension f_ε of f such that f_ε satisfies $f_\varepsilon = f$ on $[0, \rho_2]$ and the following conditions which we refer to as (F_ε):

$$\begin{aligned}
&f_\varepsilon \text{ is bounded,} && \\
&f_\varepsilon(s) \equiv \varepsilon && \text{for} \quad s \in (-\infty, -1] \\
&f_\varepsilon \in C^1(-1, 0),\ f_\varepsilon(s) \in (0, \varepsilon) \text{ and is decreasing} && \text{for} \quad s \in (-1, 0) \\
&\lim_{s \to 0^-} f_\varepsilon(s)/(|s|^{p-2}\, s) = -m && \\
&f_\varepsilon(s) < 0 && \text{for} \quad s \in (\rho_2, \infty) \\
&\int_\rho^{\rho_2} f_\varepsilon(s)\, ds > 0 && \text{for} \quad \rho \in [-1, 0].
\end{aligned}$$

Moreover,

$$\lim_{s \to (-1)^+} (f_\varepsilon(s) - \varepsilon)/(s+1)^{p-1} = 0 \qquad \text{for} \quad p > 1$$

and for $1 < p < 2$,

$$\lim_{s \to (-1)^+} f'_\varepsilon(s) = 0.$$

Since

$$f(s)/s = (f(s)/s^{p-1})\, s^{p-2}, \qquad \frac{f_\varepsilon(s) - \varepsilon}{(s+1)} = \frac{f_\varepsilon(s) - \varepsilon}{(s+1)^{p-1}}\,(s+1)^{p-2},$$

we have

$$\lim_{s \to 0} f'_\varepsilon(s) = 0, \qquad \lim_{s \to -1} f'_\varepsilon(s) = 0 \quad \text{for} \quad p > 2,$$

$$\lim_{s \to 0} f'_\varepsilon(s) = -m, \qquad \lim_{s \to -1} f'_\varepsilon(s) = 0 \quad \text{for} \quad p = 2$$

and

$$\lim_{s\to 0^+} f'_\varepsilon(s) = -\infty, \qquad \lim_{s\to -1} f'_\varepsilon(s) = 0 \quad \text{for} \quad 1<p<2.$$

Therefore, $f_\varepsilon \in C^1(-\infty, \infty)$ for $p \geqslant 2$ and $f_\varepsilon \in C^1((-\infty, \infty)\setminus\{0\})$ for $1<p<2$. Moreover, since $\lim_{s\to 0} f_\varepsilon(s)/(|s|^{p-2}\, s) = -m$, we have

$$f'_\varepsilon(s) \sim |s|^{p-2} \qquad \text{for } s \text{ near } 0.$$

Thus, we can choose $M>0$ sufficiently large such that $f_\varepsilon(s) + M\,|s|^{p-2}\, s$ is increasing on $(-\infty, \rho_2]$.

LEMMA 2.3. *Let f_ε be defined as above. Then there exists $\varepsilon_0 > 0$ such that for $0<\varepsilon<\varepsilon_0$ and $\mu>0$ sufficiently large (μ is independent of ε), there exists $v_{\mu,\varepsilon} \in C^1(\mathbb{R}^N)$, radially symmetric, which satisfies:*

$$-\operatorname{div}(|Dv_{\mu,\varepsilon}|^{p-2}\, Dv_{\mu,\varepsilon}) = \mu(f_\varepsilon(v_{\mu,\varepsilon}) - \varepsilon) \qquad \text{in } \mathbb{R}^N,$$
$$v_{\mu,\varepsilon}(0) \in (\rho_1, \rho_2),$$
$$v_{\mu,\varepsilon}(1) = -1.$$

Moreover, either $v'_{\mu,\varepsilon}(r) < 0$ for $r>0$ or $v'_{\mu,\varepsilon}(r) \equiv 0$ in $[0, r_{\mu,\varepsilon}]$ with $0 \leqslant r_{\mu,\varepsilon} < 1$, $v'_{\mu,\varepsilon}(r) < 0$ for $r > r_{\mu,\varepsilon}$, $\lim_{\varepsilon\to 0} \lim_{\mu\to\infty} \max_\Omega v_{\mu,\epsilon} = \rho_2$.

Proof. We first choose $\varepsilon>0$ such that $\tilde f_\varepsilon(s) = f_\varepsilon(s-1) - \varepsilon$ satisfies (F_1) and (F_2) (with $-m$ in (F_1) being 0 here). In fact, since f_ε satisfies (F_ε), we see that $\tilde f_\varepsilon(0) = 0$. For $\varepsilon > 0$ sufficiently small, there exists $\theta_i(\varepsilon) > 0$ $(i=1,2)$ satisfying $\theta_i(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that $\tilde f_\varepsilon(\rho_1 + 1 + \theta_1(\varepsilon)) = 0$ and $\tilde f_\varepsilon(\rho_2 + 1 - \theta_2(\varepsilon)) = 0$ and $\tilde f_\varepsilon < 0$ in $(0, \rho_1 + 1 + \theta_1(\varepsilon))$, $\tilde f_\varepsilon > 0$ in $(\rho_1 + 1 + \theta_1(\varepsilon), \rho_2 + 1 - \theta_2(\varepsilon))$, and $\tilde f_\varepsilon < 0$ in $(\rho_2 + 1 - \theta_2(\varepsilon), \infty)$. Moreover,

$$\int_0^{\rho_2+1-\theta_2(\varepsilon)} \tilde f_\varepsilon(s)\, ds = \int_0^{\rho_2+1-\theta_2(\varepsilon)} f_\varepsilon(\xi - 1)\, d\xi - \varepsilon(\rho_2 + 1 - \theta_2(\varepsilon))$$
$$= \int_{-1}^0 f_\varepsilon(s)\, ds + \int_0^{\rho_2-\theta_2(\varepsilon)} f_\varepsilon(s)\, ds - \varepsilon(\rho_2 + 1 - \theta_2(\varepsilon)).$$

Since $\lim_{\varepsilon\to 0} \theta_2(\varepsilon) = 0$, it follows that

$$\lim_{\varepsilon\to 0} \left[\int_{-1}^0 f_\varepsilon(s)\, ds + \int_0^{\rho_2-\theta_2(\varepsilon)} f_\varepsilon(s)\, ds - \varepsilon(\rho_2 + 1 - \theta_2(\varepsilon)) \right] > 0$$

as $\int_0^{\rho_2} f_\varepsilon(s)\,ds > 0$ and $\int_{-1}^0 f_\varepsilon(s)\,ds \leqslant \varepsilon$. Thus there exists $\varepsilon_0 > 0$ such that

$$\int_0^{\rho_2+1-\theta_2(\varepsilon_0)} \tilde{f}_\varepsilon(s)\,ds = \int_0^{\rho_2+1-\theta_2(\varepsilon_0)} f_\varepsilon(\xi-1)\,d\xi - \varepsilon_0(\rho_2+1-\theta_2(\varepsilon_0)) > 0.$$

Therefore, $\int_0^{\rho_2+1-\theta_2(\varepsilon)} \tilde{f}_\varepsilon(s)\,ds > 0$ for $0 < \varepsilon < \varepsilon_0$. Similarly we obtain

$$\int_\rho^{\rho_2+1-\theta_2(\varepsilon)} \tilde{f}_\varepsilon(s)\,ds > 0 \text{ for } \rho \in [0, \rho_2+1-\theta_2(\varepsilon)).$$

Here we use the facts that $\int_\rho^{\rho_2-\theta_2(\varepsilon)} f_\varepsilon(s)\,ds \geqslant \int_0^{\rho_2-\theta_2(\varepsilon)} f_\varepsilon(s)\,ds$ for $\rho \geqslant 0$ and that $\int_\rho^0 f_\varepsilon(s)\,ds \geqslant 0$ if $\rho \in [-1, 0)$. Therefore, as in the proof of Lemma 5.1 below, there exists a positive radial solution $w_{\mu,\varepsilon} \in C_0^1(B)$ of

$$-\Delta_p u = \mu \tilde{f}_\varepsilon(u) \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B,$$

where B is the unit ball in $\mathbb{R}^N$, $w_{\mu,\varepsilon}$ is a global minimizer of the functional

$$I_\varepsilon(u) = \frac{1}{p}\int_0^1 r^{N-1}\,|u'|^p\,dr - \mu\int_0^1 r^{N-1}\tilde{F}_\varepsilon(u(r))\,dr,$$

where $\tilde{F}_\varepsilon(u) = \int_0^u \tilde{f}_\varepsilon(s)\,ds$, and satisfies

$$\max w_{\mu,\varepsilon} \in (\rho_1+1+\theta_1(\varepsilon), \rho_2+1-\theta_2(\varepsilon)] \subset (\rho_1+1, \rho_2+1)$$

and $\max w_{\mu,\varepsilon} \to \rho_2+1-\theta_2(\varepsilon)$ as $\mu \to +\infty$. (Note that $\lim_{\varepsilon\to 0^+} \theta_i(\varepsilon) = 0$ for $i = 1, 2$.) The limit of $\max w_{\mu,\varepsilon}$ follows from an argument similar to that in the proof of Lemma 5.1 below.

From Lemmas 2.1 and 2.2 we have the following: $w'_{\mu,\varepsilon}(r) < 0$ for $r \in (0, 1]$ if $\max w_{\mu,\varepsilon} < \rho_2+1-\theta_2(\varepsilon)$; there exists $0 \leqslant r_{\mu,\varepsilon} < 1$ such that $w_{\mu,\varepsilon} \equiv \rho_2+1-\theta_2(\varepsilon)$ in $[0, r_{\mu,\varepsilon}]$ and $w'_{\mu,\varepsilon}(r) < 0$ for $r_{\mu,\varepsilon} < r \leqslant 1$ if $\max w_{\mu,\varepsilon} = \rho_2+1-\theta_2(\varepsilon)$. Here we use the facts that

$$\lim_{s\to 0^+} \tilde{f}_\varepsilon(s)/s^{p-1} = 0 \qquad \text{for} \quad p > 1$$

and that Lemma 2.2 still holds when $-m$ in (F_1) is equal to 0.

Set $v_{\mu,\varepsilon}(r) = w_{\mu,\varepsilon}(r) - 1$ for $r \in [0, 1]$ and

$$v_{\mu,\varepsilon}(r) = \begin{cases} -1 + (r^d - 1)\cdot d^{-1}\cdot w'_{\mu,\varepsilon}(1) & \text{for} \quad r \in (1, \infty) \quad \text{if } p \neq N, \\ -1 + \log r \cdot w'_{\mu,\varepsilon}(1) & \text{for} \quad r \in (1, \infty) \quad \text{if } p = N, \end{cases}$$

where $d = (p-N)/(p-1)$. Since $f_\varepsilon - \varepsilon = 0$ on $(-\infty, -1]$, one verifies that $v_{\mu,\varepsilon}$ is the required function. This completes the proof. ∎

DEFINITION 1. We call a function v a subsolution (supersolution) of the problem

$$-\Delta_p u = \lambda g(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

if

(i) $v \in W^{1,p}(\Omega) \cap C^0(\bar{\Omega})$,
(ii) $v \leqslant (\geqslant)\, 0$ on $\partial\Omega$, and
(iii) $\int_\Omega (|Dv|^{p-2}\, Dv\, D\phi - \lambda g(v)\, \phi)\, dx \leqslant 0\, (\geqslant 0)$ for every $\phi \in D^+(\Omega)$, where $D^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$.

We write $(1.1)_\varepsilon$ to denote the problem (1.1) with f replaced by f_ε.

COROLLARY 2.4. *Let $(\mu, v_{\mu,\varepsilon})$ be as in Lemma 2.3, and let $\alpha_{\mu,\varepsilon} \in (0, 1)$ be the unique zero of $v_{\mu,\varepsilon}$. Then for $y \in \Omega$ and $\lambda > \mu \cdot \alpha_{\mu,\varepsilon}^p \cdot \operatorname{dist}(y, \partial\Omega)^{-p}$,*

$$w_{\mu,\varepsilon}(\lambda, y; x) := v_{\mu,\varepsilon}((\lambda/\mu)^{1/p} \cdot (x - y)), \qquad x \in \Omega \tag{2.4}$$

is a subsolution of $(1.1)_\varepsilon$.

Proof. We omit the subscripts μ and ε on w and v in the following for simplicity. The function $w(\lambda, y) \in C^1(\Omega)$ satisfies

$$-\operatorname{div}(|Dw|^{p-2}\, Dw) = \lambda(f_\varepsilon(w) - \varepsilon) \quad \text{in } \Omega;$$

hence $\int_\Omega (|Dw|^{p-2}\, Dw\, D\phi - \lambda f_\varepsilon(w)\, \phi)\, dx \leqslant 0$ for all $\phi \in D^+(\Omega)$. Since $w(\lambda, y) < 0$ on $\partial\Omega$ for $\lambda > \mu\alpha^p \cdot \operatorname{dist}(y, \partial\Omega)^{-p}$, $w(\lambda, y)$ satisfies the definition of subsolution. This proves the corollary. ∎

Next we prove an appropriate version of the sweeping principle of Serrin.

PROPOSITION 2.5. *Let $\varepsilon > 0$ and f_ε be as above, let u with $\sup_\Omega u < \rho_2$ be a supersolution of the problem $-\Delta_p u = f_\varepsilon(u)$, $u = 0$ on $\partial\Omega$, and let*

$$A = \{v_t \colon \sup_\Omega v_t < \rho_2,\ t \in [0, 1]\}$$

be a family of subsolutions of

$$-\Delta_p v = \lambda(f_\varepsilon(v) - \varepsilon) \quad \textit{in } \Omega, \qquad v = 0 \quad \textit{on } \partial\Omega$$

satisfying $v_t < 0$ *on* $\partial\Omega$ *for all* $t \in [0, 1]$. *If*

(i) $t \mapsto v_t$ *is continuous relative to the* $\|\cdot\|_0$*-norm,*
(ii) $u \geqslant v_0$ *in* $\bar{\Omega}$, *and*
(iii) $u \not\equiv v_t$, *for all* $t \in [0, 1]$,

then $u \geqslant v_t$ *in* $\bar{\Omega}$ *for all* $t \in [0, 1]$.

Proof. Set $E = \{t \in [0, 1];\ u \geqslant v_t \text{ in } \bar{\Omega}\}$. By (ii), E is nonempty. Moreover, E is closed. By the conditions on f_ε, there exists $M > 0$ such that

$$g_\varepsilon(s) := f_\varepsilon(s) + M\,|s|^{p-2}\,s$$

is strictly increasing on $[0, \rho_2]$ since $\lim_{s\to 0} |f_\varepsilon(s)|/(|s|^{p-2}\,s) < \infty$.

As $u > v_t$ on $\partial\Omega$ for all $t \in [0, 1]$, it follows that there exists $\tau > 0$ independent of t such that $u \geqslant v_t + \tau$ on $\partial\Omega$. Let $w = v_t + \tau$. We choose $\tau > 0$ such that

$$[M(|v_t+\tau|^{p-2}\,(v_t+\tau) - |v_t|^{p-2}\,v_t) - \varepsilon] \leqslant 0$$

in Ω for all $t \in [0, 1]$. Then for $t \in E$, we have

$$\begin{aligned} -\Delta_p u + \lambda M\,|u|^{p-2}\,u &\geqslant \lambda f_\varepsilon(u) + \lambda M\,|u|^{p-2}\,u \\ &\geqslant \lambda f_\varepsilon(v_t) + \lambda M\,|v_t|^{p-2}\,v_t \\ &\geqslant -\Delta_p v_t + \lambda\varepsilon + \lambda M\,|v_t|^{p-2}\,v_t \\ &\geqslant -\Delta_p w + \lambda M\,|w|^{p-2}\,w. \end{aligned}$$

The weak comparison principle [5] implies that $u \geqslant v_t + \tau$ in Ω. Thus, $u \geqslant v_t + \tau$ in $\bar{\Omega}$. Since $t \mapsto v_t$ is continuous with respect to the $\|\cdot\|_0$-norm, this shows that E is also open. Hence $E = [0, 1]$. ∎

Remark 2.6. (1) It is easily seen that Proposition 2.5 is valid for every small $\varepsilon > 0$.

(2) We can obtain a similar sweeping principle when u with $\sup_\Omega u < \rho_2$ is a subsolution of (1.1) and $A = \{v_t;\ \sup_\Omega v_t \leqslant \rho_2,\ t \in [0, 1]\}$ is a family of supersolutions of

$$-\Delta_p v = \lambda(f(v) + \varepsilon) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega$$

satisfying $v_t > 0$ on $\partial\Omega$ by reversing the remaining inequalities.

3. ASYMPTOTIC BEHAVIOUR OF POSITIVE SOLUTIONS OF (1.1) WHEN λ IS LARGE

In this section we shall study the asymptotic behaviour of the positive solutions of (1.1) when the parameter λ is large.

In the following we always assume that Ω is a bounded connected smooth domain in $\mathbb{R}^N$. We denote by $t(x)$ the distance from $x \in \Omega$ to the boundary $\partial\Omega$ and by $s(x)$ the point of $\partial\Omega$ ($s(x) = x$ if $x \in \partial\Omega$) which is closest to x (which is uniquely defined if x is close enough to $\partial\Omega$). We choose $\delta_0 > 0$ so small that the boundary strip $\{x \in \Omega : 0 < t(x) < \delta_0\}$ is covered (and only covered) by the straight lines in the inner normal direction $n_{s(x)}$ and emanating from $s(x)$.

Let $x^* \in \Omega$ and define $\lambda^* = \mu \operatorname{dist}(x^*, \partial\Omega)^{-p}$ and for $\lambda > \lambda^*$ let $z_\lambda = w(\lambda, x^*)$, where μ and w are as defined in Corollary 2.4.

LEMMA 3.1. *Let f_ε satisfy* (F_1)–(F_3) *and* (F_ε). *Then*

(i) *for $\lambda > \lambda^*$, $(1.1)_\varepsilon$ possesses a maximal solution $u_\lambda \in [z_\lambda, \rho_2]$,*

(ii) *there exist $\lambda^{**} > \lambda^*$, $c > 0$, and $\tau \in (\rho_1, \rho_2)$ such that for $\lambda > \lambda^{**}$ every solution $u_\lambda \in [z_\lambda, \rho_2]$ of* (1.1) *satisfies*

$$u_\lambda(x) > \min(c\lambda^{1/p} \operatorname{dist}(x, \partial\Omega), \tau) \qquad \textit{for all} \quad x \in \Omega. \tag{3.1}$$

Proof. By Corollary 2.4, for $\lambda > \lambda^*$ we have z_λ is a subsolution of $(1.1)_\varepsilon$ and $z_\lambda < \rho_2$. Since ρ_2 is a supersolution of $(1.1)_\varepsilon$ and there exists $M > 0$ such that $f_\varepsilon(s) + M\,|s|^{p-2}\,s$ is strictly increasing in $(\min_\Omega z_\lambda, \rho_2]$, by a monotone method as in [5, 17, 40], there is a maximal solution $u_\lambda \in [z_\lambda, \rho_2]$ for $\lambda > \lambda^*$. This proves (i).

Since Ω satisfies a uniform interior sphere condition, there exists $\eta_0 > 0$ such that $\Omega = \bigcup\{B(x, \eta); x \in \Omega_\eta\}$ for $\eta \in (0, \eta_0]$, where $\Omega_\eta = \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > \eta\}$. Set

$$\lambda^{**} = \max(\lambda^*, \mu\alpha^p\eta_0^{-p}), \qquad c = \mu^{-1/p} \inf\{(\alpha - r)^{-1}\, v(r);\ r \in [0, \alpha)\}, \qquad \tau = v(0)$$

with μ, v, and α as in Corollary 2.4. Note that $c > 0$, since $v > 0$ on $[0, \alpha)$ and $v'(\alpha) < 0$.

Let u_λ be a solution of $(1.1)_\varepsilon$, $\lambda > \lambda^{**}$, and $u_\lambda \in [z_\lambda, \rho_2]$. Since for $\lambda > \lambda^{**}$, $\Omega_{\alpha(\mu/\lambda)^{1/p}}$ is arcwise connected (note that $x^* \in \Omega_{\alpha(\mu/\lambda)^{1/p}}$) and since $w(\lambda, y)$ is a subsolution for $y \in \Omega_{\alpha(\mu/\lambda)^{1/p}}$, with $w(\lambda, y) < 0$ on $\partial\Omega$, by Proposition 2.5 we obtain

$$u_\lambda > w(\lambda, y) \qquad \text{in } \Omega \quad \text{for all } y \in \Omega_{\alpha(\mu/\lambda)^{1/p}}.$$

(Note that $w(\lambda, y)$ is a subsolution of the problem

$$-\Delta_p v = \lambda(f_\varepsilon(v) - \varepsilon) \text{ in } \Omega, \qquad v = 0 \text{ on } \partial\Omega.)$$

Hence, an argument similar to that in [6] implies

$$u_\lambda(x) > c\lambda^{1/p} \operatorname{dist}(x, \partial\Omega) \qquad \text{for all} \quad x \in \Omega \backslash \Omega_{\alpha(\mu/\lambda)^{1/p}},$$

$$u_\lambda(x) > \tau \qquad \text{for all} \quad x \in \Omega_{\alpha(\mu/\lambda)^{1/p}},$$

which completes the proof. ∎

Remark 3.2. It follows from Eq. (3.1) that the maximal solution u_λ is positive for $\lambda > \lambda^{**}$, and that $\max u_\lambda \in (\rho_1, \rho_2]$ for λ sufficiently large. This implies that u_λ is a positive solution of (1.1).

Let ψ be the eigenfunction corresponding to the first eigenvalue ν_1 of

$$-\Delta_p v = \nu\, |v|^{p-2}\, v \text{ in } B, \qquad v = 0 \quad \text{on } \partial B,$$

where B denotes the unit ball in $\mathbb{R}^N$. Let ψ be normalized so that $\max \psi = 1$. It is well known that $\psi > 0$ in B, ψ is radially symmetric, and $\psi(0) = 1$.

LEMMA 3.3. *Let u satisfy $-\Delta_p u = \lambda f_\varepsilon(u)$ in an open set $\Omega' \subset \Omega$. Let $\sigma, \varepsilon > 0$ (with ε sufficiently small) be such that $f_\varepsilon(s) - \varepsilon \geqslant \sigma(s-a)^{p-1}$ for $s \in [a, b]$. Suppose that $u(x) > a$ for $x \in \Omega'$. If $x_0 \in (\Omega')_{(\nu_1/(\sigma\lambda))^{1/p}}$, then $u(x_0) > b$.*

Proof. Set $\theta(x_0, \lambda, t; x) = a + t\psi(((\sigma\lambda)/\nu_1)^{1/p}(x - x_0))$ for $x \in \tilde{B}$ and $t \in [0, b-a]$, where $\tilde{B}$ is the ball $B(x_0, (\nu_1/\sigma\lambda)^{1/p})$. We claim that the set $\{\theta(x_0, \lambda, t); t \in [0, b-a]\}$ is a family of subsolutions of the problem

$$-\operatorname{div}(|Dv|^{p-2}\, Dv) = \lambda(f_\varepsilon(v) - \varepsilon) \quad \text{in } \tilde{B}, \qquad v = u \quad \text{on } \partial\tilde{B} \tag{3.2}$$

and the closure of $\tilde{B}$ is contained in Ω'. Then, by a method similar to that in the proof of Proposition 2.5, we would obtain $u(x_0) > b$. So it remains to show that $\theta(x_0, \lambda, t)$ is a subsolution of Eq. (3.2). This can be seen from a routine calculation and the fact that $u > a = \theta(x_0, \lambda, t)$ on $\partial\tilde{B}$. ∎

LEMMA 3.4. *Let f_ε satisfy* (F_1)–(F_3) *and* (F_ε). *For every $\tilde{\delta} > 0$ there is a $c(\tilde{\delta}) > 0$ such that for all solutions u_λ of $(1.1)_\varepsilon$, $\lambda > \lambda^{**}$, and $u_\lambda \in [z_\lambda, \rho_2]$, the following inequality holds*

$$u_\lambda(x) > \min(c(\tilde{\delta})\, \lambda^{1/p} \operatorname{dist}(x, \partial\Omega), \rho_2 - \tilde{\delta}) \qquad \textit{for all} \quad x \in \Omega,$$

where λ^{**} and z_λ are as in Lemma 3.1.

Proof. If $\rho_2-\tilde{\delta}<\tau$, we are done with $c(\tilde{\delta})=c$ as in Lemma 3.1. Otherwise, by (F_1), there exist σ, $\varepsilon>0$ (ε sufficiently small) such that $f_\varepsilon(s)-\varepsilon>\sigma(s-\tau)^{p-1}$ for all $s\in[\tau,\rho_2-\tilde{\delta}]$ since $f_\varepsilon(s)>0$ in $[\tau,\rho_2-\tilde{\delta}]$. (Note that σ depends on ε.) Let v_1 be as in Lemma 3.3. Using Lemma 3.3 with $\Omega'=\Omega_{k\lambda^{-1/p}}$, $k=c^{-1}\tau$, and Lemma 3.1, since $(\Omega')_{(v_1/(\sigma\lambda))^{1/p}}=\Omega_{((v_1/\sigma)^{1/p}+k)\,\lambda^{-1/p}}$, we obtain

$$u(x)>\rho_2-\tilde{\delta}\qquad\text{for all}\quad x\in\Omega_{((v_1/\sigma)^{1/p}+k)\,\lambda^{-1/p}}.$$

By (3.1) we have

$$u(x)>c(\tilde{\delta})\,\lambda^{1/p}\operatorname{dist}(x,\partial\Omega)\qquad\text{for all}\quad x\in\Omega\setminus\Omega_{((v_1/\sigma)^{1/p}+k)\,\lambda^{-1/p}}\tag{3.3}$$

with $c(\tilde{\delta})=\min\{c,\tau((v_1/\sigma)^{1/p}+k)^{-1}\}$. This completes the proof. ∎

Now we consider the problem

$$-(|y'|^{p-2}\,y')'=f(y),\qquad y(0)=0,\qquad y(\infty)=\rho_2.\tag{3.4}$$

By arguments similar to those in the proof of Lemma 3.1 in [24], we see that Eq. (3.4) has a unique solution $z_0(t)$ when f satisfies (F_1)–(F_2). Moreover, z_0 satisfies either

(A) $z_0>0$, $z_0'>0$ in $(0,\infty)$ and $\lim_{t\to\infty}z_0(t)=\rho_2$, or

(B) $z_0>0$, there exists $\bar{t}>0$ such that $z_0'>0$ in $(0,\bar{t})$, $z_0'(\bar{t})=0$, and $z_0\equiv\rho_2$ in $[\bar{t},\infty)$.

Remark 3.5. Case (B) cannot occur when f satisfies (F_3). In fact, from the first integral of Eq. (3.4), we have

$$|z_0'(t)|^p+p'F(z_0(t))\equiv C,\qquad t\in(0,\infty),$$

where $1/p+1/p'=1$. Therefore,

$$|z_0'|^p=p'(F(\rho_2)-F(z_0)).$$

Since $F(\rho_2)>F(s)$ for $0<s<\rho_2$, we have

$$\int_0^{z_0(t)}(F(\rho_2)-F(s))^{-1/p}\,ds=(p')^{1/p}\,t.$$

Since $\int_0^{\rho_2}(F(\rho_2)-F(s))^{-1/p}\,ds=\infty$ when f satisfies (F_3), (B) cannot occur.

If $x\in\Omega$ and x is near $\partial\Omega$, x can be uniquely written in the form $x=s+tn_s$ where $s=s(x)\in\partial\Omega$, $n_s=n_{s(x)}$ denotes the inward unit normal vector to $\partial\Omega$ at $s(x)$, and $t = t(x)$ is small and positive. We will make frequent use of these coordinates. If $\lambda > 0$, define $\eta_\lambda(x) = z_0(\lambda^{1/p} t)$ if x is near $\partial\Omega$ and $\eta_\lambda(x) = \rho_2$ otherwise.

PROPOSITION 3.6. *Let f satisfy* (F$_1$)–(F$_3$). *For every small $\tilde{\varepsilon} > 0$, there is $\bar{\lambda} = \bar{\lambda}(\tilde{\varepsilon}) > \lambda^{**}$ such that if $\lambda \geqslant \bar{\lambda}$ and $u_\lambda \in [z_\lambda, \rho_2]$ is a positive solution of* (1.1), *then*

$$(1 - \tilde{\varepsilon})\, \eta_\lambda \leqslant u_\lambda \leqslant (1 + \tilde{\varepsilon})\, \eta_\lambda. \tag{3.5}$$

Proof. It is clear that if u_λ is a positive solution of (1.1) with $\|u_\lambda\|_\infty \leqslant \rho_2$, then u_λ is a solution of $(1.1)_\varepsilon$ for any small $\varepsilon > 0$. We also know that $\|u_\lambda\|_\infty < \rho_2$ by the strong maximum principle in [44] (this can be obtained by the argument similar to that in the proof of Proposition 4.2). For convenience, we omit the subscript λ on u_λ below.

By Lemma 3.4, it remains to prove the result for points whose distance from $\partial\Omega$ is of order $\lambda^{-1/p}$. To prove this, we construct sub- and super-solutions. The key step in the proof below is to establish the sweeping out results.

Near $\partial\Omega$, we use the s, t coordinates. In these variables,

$$\begin{aligned} \Delta_p u = {} & (|u'_t|^{p-2}\, u'_t)'_t + b(s,t)\, |u'_t|^{p-2}\, u'_t + [(|u'_t + u'_s|^{p-2}\, u'_t)'_t - (|u'_t|^{p-2}\, u'_t)'_t] \\ & + b(s,t)[|u'_t + u'_s|^{p-2}\, u'_t - |u'_t|^{p-2}\, u'_t] + \text{terms involving } s \text{ derivatives}, \end{aligned}$$

where $u'_t = \frac{\partial u}{\partial t}$, $u'_s = \frac{\partial u}{\partial s}$. By the conditions imposed on f, there exists $M > 0$ such that $g(s) := f(s) + M s^{p-1}$ is increasing in $(0, \rho_2]$.

If $\bar{\alpha} < z'_0(0)$ but is close, using the first integral of Eq. (3.4) we easily prove that the solution $\tilde{z}$ of Eq. (3.4) with $\tilde{z}(0) = 0$, $\tilde{z}'(0) = \bar{\alpha}$, first increases to a number near ρ_2 but less than ρ_2 and then decreases to zero (see [24]). Hence there is $\tilde{l}$ near ρ_2 and $\tilde{t} > 0$ such that $\tilde{z}(\tilde{t}) = \tilde{l}$, $\tilde{z}'(\tilde{t}) = 0$. Since $(|\tilde{z}'|^{p-2}\, \tilde{z}')' = -f(\tilde{z}(\tilde{t})) \neq 0$, $\tilde{z}'(\tilde{t})$ changes sign at $\tilde{t}$. Hence if μ is close to 1 and β is small, the solution $\bar{z}$ of

$$-(|x'|^{p-2}\, x')' - \beta\, |x'|^{p-2}\, x' + M x^{p-1} = \mu g(x(t)), \qquad x(0) = 0,\ x'(0) = \bar{\alpha} \tag{3.6}$$

increases until $\bar{t}$ where $\bar{z}'(\bar{t}) = 0$ and $\bar{z}(\bar{t})$ is close to ρ_2 but less than ρ_2. Define

$$\tilde{\eta}_\lambda(x) = \begin{cases} \bar{z}(\lambda^{1/p} t), & \text{if } x \text{ is close to } \partial\Omega \text{ and } 0 \leqslant t \leqslant \lambda^{-1/p}\, \bar{t}, \\ \bar{z}(\bar{t}), & \text{otherwise}, \end{cases}$$

where $x = s + tn_s$ if x is near $\partial\Omega$. (Thus $\tilde{\eta}_\lambda$ is constant except near $\partial\Omega$.) Suppose we can show that, if λ is large and $u \in [z_\lambda, \rho_2]$ is a positive solution of (1.1), then $u \geqslant \tilde{\eta}_\lambda$. Since $\tilde{z}$ is close to z_0 on compact intervals if $\bar{\alpha}$ is near $z_0''(0)$, μ is near 1, and β is small, then we obtain that $u \geqslant (1-\tilde{\varepsilon})\,\eta_\lambda$ for λ large. This will prove half of Proposition 3.6.

By choosing $\beta < 0$ and $\mu < 1$, we have $\tilde{\eta}_\lambda \in C^1$. Moreover, there is $c \in (0, 1)$ such that $u \geqslant \tilde{\eta}_{c\lambda}$ for λ large by the proof of Lemma 3.4. Now we show that

$$u \geqslant \tilde{\eta}_{j\lambda} \qquad \text{for} \quad j \in [c, 1]. \tag{3.7}$$

A sweeping principle similar to that of Proposition 2.5 will give (3.7). We only need to show that if $j \in [c, 1]$ and $u \geqslant \tilde{\eta}_{j\lambda}$ in Ω, there exists $\xi > 0$ such that

$$u - \tilde{\eta}_{j\lambda} \geqslant \xi e_0 \qquad \text{in } \Omega, \tag{3.8}$$

where $e_0(x)$ is the unique positive solution of the problem

$$-\Delta_p e_0 = 1 \quad \text{in } \Omega, \qquad e_0 = 0 \quad \text{on } \partial\Omega.$$

Our conclusion (3.7) will be obtained from the continuity of $\tilde{\eta}_{j\lambda}$ and $\partial\tilde{\eta}_{j\lambda}/\partial\nu$ under the norms $\|\cdot\|_{L^\infty(\Omega)}$, $\|\cdot\|_{L^\infty(\partial\Omega)}$ (respectively) with respect to $j \in [c, 1]$ where $\nu(x) = -n_{s(x)}$ is the outward normal vector to $\partial\Omega$ at x (note that $s(x) = x$).

In fact, since $u(x) = 0$, $\tilde{\eta}_{j\lambda}(x) = 0$ for $x \in \partial\Omega$, and $g(s)$ is increasing for $s \in (0, \rho_2]$, it follows from the strong maximum principle in [44] that

$$\frac{\partial u}{\partial \nu} < 0, \quad \frac{\partial \tilde{\eta}_{j\lambda}}{\partial \nu} \leqslant -(c\lambda)^{1/p}\,\bar{z}'(0) < 0 \qquad \text{on } \partial\Omega.$$

Thus, there exists a one-sided neighbourhood Λ_λ of $\partial\Omega$ contained in Ω (we may choose $\Lambda_\lambda \subset \{x \in \Omega : 0 < t < \bar{t}\lambda^{-1/p}\} \subset \{x \in \Omega : 0 < t < \delta_0\}$) such that $\partial u/\partial(-n_{s(x)}) < 0$ and $\partial\tilde{\eta}_{j\lambda}/\partial(-n_{s(x)}) < 0$ for $x \in \Lambda_\lambda$. For $x \in \Lambda_\lambda$,

$$\begin{aligned}
&-\operatorname{div}(|Du|^{p-2}\,Du) - \{-\operatorname{div}(|D\tilde{\eta}_{j\lambda}|^{p-2}\,D\tilde{\eta}_{j\lambda})\} \\
&\quad = \lambda(f(u) - j\mu f(\tilde{\eta}_{j\lambda})) + j\lambda[b(s,t)(j\lambda)^{-1/p} - \beta](\bar{z}'((j\lambda)^{1/p}\,t))^{p-1} \\
&\qquad + j\lambda(1-\mu)\,M\tilde{\eta}_{j\lambda}^{p-1}.
\end{aligned}$$

Since the first term on the right hand side of the above identity is 0 on $\partial\Omega$ and the second term is positive (for λ sufficiently large), there is a one-sided neighbourhood $\Lambda_\lambda^* \subset \Lambda_\lambda$ of $\partial\Omega$ such that the right hand side of the above identity is nonnegative for $x \in \Lambda_\lambda^*$. On the other hand, since $u - \tilde{\eta}_{j\lambda} = 0$ on $\partial\Omega$ and

$$-\operatorname{div}(|Du|^{p-2} Du) - \{-\operatorname{div}(|D\tilde{\eta}_{j\lambda}|^{p-2} D\tilde{\eta}_{j\lambda})\} = -L(u - \tilde{\eta}_{j\lambda})$$

(see [23]) where L is a uniformly elliptic operator in Λ_λ^*, by the maximum principle for L, we have $(\partial/\partial\nu)(u - \tilde{\eta}_{j\lambda}) < 0$ on $\partial\Omega$. This implies that there exist $\xi_1 > 0$ and $\Lambda_\lambda^{**} \subset \Lambda_\lambda^*$ such that

$$u(x) - \tilde{\eta}_{j\lambda}(x) \geqslant \xi_1 e_0(x) \qquad \text{for} \quad x \in \Lambda_\lambda^{**}. \tag{3.9}$$

Choose a smooth domain $\Omega_\lambda \subset \Omega$ with $\partial\Omega_\lambda \subset \Lambda_\lambda^{**}$. Then, there exists $\tau_\lambda^1 > 0$ such that $u \geqslant \tilde{\eta}_{j\lambda} + \tau_\lambda^1$ on $\partial\Omega_\lambda$. Defining $\Omega_j = \{x = s + tn_s \in \Omega;\ t > (j\lambda)^{-1/p}\,\bar{t}\}$, without loss of generality we may assume $\Omega_j \subset\subset \Omega_\lambda$.

We claim that

$$u(x) > \tilde{\eta}_{j\lambda}(x) \qquad \text{for} \quad x \in \overline{\Omega_j}. \tag{3.10}$$

In fact, we know that $\tilde{\eta}_{j\lambda} \equiv \bar{z}(\bar{t})$ in $\overline{\Omega_j}$. Suppose that there exists $x_0 \in \overline{\Omega_j}$ such that $u(x_0) = \bar{z}(\bar{t})$. We consider two cases: (i) $x_0 \in \Omega_j$ and (ii) $x_0 \in \partial\Omega_j$. In case (i), letting $w = u - \bar{z}(\bar{t})$, we have $w \geqslant 0$ in Ω_j and w satisfies

$$-\Delta_p w = \lambda f(u) > 0 \qquad \text{in} \quad \Omega_j$$

since $u \geqslant \bar{z}(\bar{t}) > \rho_1$ in Ω_j. The strong maximum principle in [17] implies

$$w \equiv 0, \qquad \text{that is} \quad u \equiv \bar{z}(\bar{t}) \text{ in } \Omega_j.$$

This is a contradiction since $f(\bar{z}(\bar{t})) \neq 0$. In case (ii), we know $\nabla(u - \tilde{\eta}_{j\lambda})(x_0) = 0$ since x_0 is a minimum point of $u - \tilde{\eta}_{j\lambda}$. But also $u - \tilde{\eta}_{j\lambda} = w$ in $\overline{\Omega_j}$ and the Hopf type of maximum principle (see [17]) implies $\frac{\partial w}{\partial \nu}(x_0) < 0$, where ν is the outward normal to $\partial\Omega_j$ ($\partial\Omega_j$ is smooth for λ large since $\partial\Omega$ is smooth). This is clearly impossible. Thus, our claim (3.10) holds.

Now we consider the domain $\Omega_\lambda \setminus \overline{\Omega_j}$. It is clear that there exists $\tau_\lambda^{(2)} > 0$ such that

$$u - \tilde{\eta}_{j\lambda} \geqslant \tau_\lambda^{(2)} \qquad \text{on} \quad \partial(\Omega_\lambda \setminus \bar{\Omega}_j).$$

We also have, for $x \in \Omega_\lambda \setminus \overline{\Omega_j}$,

$$\begin{aligned}
&-\operatorname{div}(|Du|^{p-2}\,Du)+\lambda M u^{p-1}-\{-\operatorname{div}(|D\tilde{\eta}_{j\lambda}|^{p-2}\,D\tilde{\eta}_{j\lambda})+\lambda M\tilde{\eta}_{j\lambda}^{p-1}\}\\
&\quad=\lambda(g(u)-g(\tilde{\eta}_{j\lambda}))+\lambda(1-j\mu)\,f(\tilde{\eta}_{j\lambda})\\
&\qquad+j\lambda[b(s,t)(j\lambda)^{-1/p}-\beta](\bar{z}'((j\lambda)^{1/p}t))^{p-1}+\lambda j(1-\mu)\,M\tilde{\eta}_{j\lambda}^{p-1}\\
&\quad=\lambda(g(u)-g(\tilde{\eta}_{j\lambda}))+\lambda(1-j\mu)\left[f(\tilde{\eta}_{j\lambda})+\frac{j(1-\mu)}{1-j\mu}\,M\tilde{\eta}_{j\lambda}^{p-1}\right]\\
&\qquad+j\lambda[b(s,t)(j\lambda)^{-1/p}-\beta](\bar{z}'((j\lambda)^{1/p}\,t))^{p-1}.
\end{aligned}$$

Define $m_j(s)=f(s)+\frac{j(1-\mu)}{1-j\mu}\,Ms^{p-1}$. Since $\frac{j(1-\mu)}{1-j\mu}\geqslant\tilde{\theta}>0$ for $j\in[c,1]$, if we choose M sufficiently large, we have that m_j is also strictly increasing in $(0,\rho_2]$ for all $j\in[c,1]$. Thus, $m_j(\tilde{\eta}_{j\lambda})\geqslant 0$ in Ω. Since g is strictly increasing in $(0,\rho_2]$, we see that $g(u)\geqslant g(\tilde{\eta}_{j\lambda})$ in $\bar{\Omega}_\lambda$. On the other hand, since $0<\mu<1$, $c\leqslant j\leqslant 1$, and $\beta<0$, there exists $\tilde{\xi}>0$ (depending upon μ, λ, M) such that

$$\lambda(1-j\mu)\,m_j(\tilde{\eta}_{j\lambda})+j\lambda[b(s,t)(j\lambda)^{-1/p}-\beta](\bar{z}'((j\lambda)^{1/p}\,t))^{p-1}\geqslant\tilde{\xi}\qquad\text{in}\quad\overline{\Omega_\lambda\setminus\overline{\Omega_j}}.$$

Similar arguments to those in the proof of Proposition 2.5 imply that we can choose $\tau_\lambda>0$ so that $u\geqslant\tilde{\eta}_{j\lambda}+\tau_\lambda$ in $\overline{\Omega_\lambda\setminus\overline{\Omega_j}}$. This implies that there exists $\xi_2>0$ such that

$$u(x)-\tilde{\eta}_{j\lambda}(x)\geqslant\xi_2 e_0(x)\qquad\text{for}\quad x\in\overline{\Omega_\lambda}. \tag{3.11}$$

Our claim (3.8) is obtained by choosing $\xi=\min\{\xi_1,\xi_2\}$. Hence, (3.7) holds.

To prove the estimate in the opposite direction, we use another family of functions. If $\bar{\alpha}_1>z_0'(0)$, it is easy to show from the first integral that the solution $\tilde{z}_1$ of Eq. (3.4) such that $\tilde{z}_1(0)=0$, $\tilde{z}_1'(0)=\bar{\alpha}_1$, increases till it hits $y=\rho_2$. Once again, by continuous dependence, the solution $\bar{z}_1$ of Eq. (3.6) such that $\bar{z}_1(0)=0$, $\bar{z}_1'(0)=\bar{\alpha}_1$ increases till it hits $y=\rho_2$ at $t=\bar{t}_1$, provided μ is near 1 and β is small. We define

$$\bar{\eta}_\lambda(x)=\begin{cases}\bar{z}_1(\lambda^{1/p}\,t), & \text{if}\quad 0\leqslant t\leqslant\lambda^{-1/p}\,\bar{t}_1,\\ \rho_2, & \text{otherwise.}\end{cases}$$

By choosing $\mu>1$ and $\beta>0$ and using arguments similar to the above, we obtain

$$u\leqslant\bar{\eta}_{j\lambda}\qquad\text{in }\Omega,\quad\text{for }\ j\in[1,c] \tag{3.12}$$

provided it is possible to choose $c>1$ such that $u \leqslant \bar{\eta}_{c\lambda}$ for λ large and all positive solutions $u \in [z_\lambda, \rho_2]$ of (1.1) with $\max_\Omega u < \rho_2$. (Note that $\bar{\eta}_{j\lambda} \in W_0^{1,p}(\Omega) \cap C^0(\bar{\Omega})$.)

We only need to show that if $j \in [1, c]$ and $\bar{\eta}_{j\lambda} \geqslant u$ in Ω, there exists $\xi > 0$ such that

$$\bar{\eta}_{j\lambda} - u \geqslant \xi e_0 \qquad \text{in } \Omega, \tag{3.13}$$

where e_0 is as above. In fact, since $\max_\Omega u < \rho_2$, it follows that $\bar{\eta}_{j\lambda} > u$ in $\overline{\Omega_j}$, where $\Omega_j = \{x = s + tn_s \in \Omega : t > (j\lambda)^{-1/p}\,\bar{t}_1\}$. Also, by arguments similar to those in the proof of the left inequality of (3.5), there is a one-sided neighbourhood $\hat{\Lambda}_\lambda$ of $\partial\Omega$ contained in Ω such that

$$\bar{\eta}_{j\lambda} - u \geqslant \xi_1 e_0(x) \qquad \text{for} \quad x \in \hat{\Lambda}_\lambda, \ \text{where } \xi_1 > 0.$$

Choose a smooth domain $\Omega_\lambda \subset \Omega$ such that $\partial\Omega_\lambda \subset \hat{\Lambda}_\lambda$. Then, there exists $\tau_\lambda > 0$ such that $\bar{\eta}_{j\lambda} \geqslant u + \tau_\lambda$ on $\partial\Omega_\lambda$. Without loss of generality, we assume that $\Omega_j \subset\subset \Omega_\lambda$. It is clear that $\bar{\eta}_{j\lambda} > u$ on $\partial\Omega_j$. Now, for $x \in \Omega_\lambda \setminus \overline{\Omega_j}$,

$$\begin{aligned}
&-\operatorname{div}(|D\bar{\eta}_{j\lambda}|^{p-2} D\bar{\eta}_{j\lambda}) + \lambda M \bar{\eta}_{j\lambda}^{p-1} - \{-\operatorname{div}(|Du|^{p-2} Du) + \lambda M u^{p-1}\} \\
&\quad = \lambda(g(\bar{\eta}_{j\lambda}) - g(u)) + \lambda(j\mu - 1)\left[f(\bar{\eta}_{j\lambda}) + \frac{j(\mu-1)}{j\mu - 1} M \bar{\eta}_{j\lambda}^{p-1} \right] \\
&\qquad + j\lambda[\beta - b(s,t)(j\lambda)^{-1/p}](\bar{z}_1'((j\lambda)^{1/p}\, t))^{p-1}
\end{aligned}$$

provided $\mu > 1$ and $\beta > 0$. Let $m_j(s)$ be as above. Since $\frac{j(\mu-1)}{j\mu-1} > \frac{\mu-1}{\mu}$ for $j \in [1, c]$, for M sufficiently large, we have that m_j is also strictly increasing on $(0, \rho_2]$ for all $j \in [1, c]$. Thus, $m_j(\bar{\eta}_{j\lambda}) \geqslant 0$ in Ω. Since g is strictly increasing on $(0, \rho_2]$, we see that $g(\bar{\eta}_{j\lambda}) \geqslant g(u)$ in $\bar{\Omega}_\lambda$. Since $\mu > 1$, $1 \leqslant j \leqslant c$, and $\beta > 0$, there exists $\tilde{\xi} > 0$ (depending on μ, λ, M) such that

$$\lambda(j\mu - 1)\, m_j(\bar{\eta}_{j\lambda}) + j\lambda[\beta - b(s,t)(j\lambda)^{-1/p}](\bar{z}_1'((j\lambda)^{1/p}\, t))^{p-1} \geqslant \tilde{\xi} \quad \text{in } \overline{\Omega_\lambda \setminus \overline{\Omega_j}}\}.$$

(Note that $\bar{\eta}_{j\lambda}'((j\lambda)^{1/p}\, t)$ has a discontinuity when $t = (j\lambda)^{-1/p}\,\bar{t}_1$; however, the presence of a jump of $\bar{\eta}_{j\lambda}'((j\lambda)^{1/p}\, t)$ here is not a difficulty since $\bar{\eta}_1'((j\lambda)^{1/p}\, t) \geqslant \hat{\alpha} > 0$ in $\Omega \setminus \overline{\Omega_j}$.) Similar arguments to those in the proof of Proposition 2.5 imply that we may choose $\tau_\lambda > 0$ so that $\bar{\eta}_{j\lambda} \geqslant u + \tau_\lambda$ in $\overline{\Omega_\lambda \setminus \overline{\Omega_j}}$. This implies that there exists $\xi_2 > 0$ such that

$$\bar{\eta}_{j\lambda}(x) - u(x) \geqslant \xi_2 e_0(x) \qquad \text{for} \quad x \in \Omega_\lambda.$$

Thus, (3.13) is obtained by choosing $\xi = \min\{\xi_1, \xi_2\}$. This also implies (3.12).

Now we show that it is possible to choose $c > 1$ such that $u \leqslant \bar{\eta}_{c\lambda}$ for λ large and all positive solutions u in $(0, \rho_2]$ of (1.1). It is easy to see that this reduces to showing that there is $K_0 > 0$ such that $u(x) \leqslant K_0 \lambda^{1/p} t$ if u is a positive solution of (1.1), x is near $\partial\Omega$, and λ is large. Obviously, it suffices to prove the result for $t \leqslant K_1 \lambda^{-1/p}$. For arbitrary $x_0 \in \partial\Omega$, let $X = \lambda^{1/p}(x - x_0)$ and $\tilde{u}(X) = u(x)$; then

$$-\operatorname{div}(|D\tilde{u}|^{p-2} D\tilde{u}) = f(\tilde{u}) \quad \text{in } \tilde{\Omega}_\lambda, \qquad \tilde{u} = 0 \quad \text{on } \partial\tilde{\Omega}_\lambda,$$

where $\tilde{\Omega}_\lambda = \{X\colon \lambda^{-1/p} X + x_0 \in \Omega\}$. By a blow-up argument as in [7], the stretching only flattens the boundary as $\lambda \to \infty$. Since $0 \in \partial\tilde{\Omega}_\lambda$ and $\|\tilde{u}\|_\infty \leqslant \rho_2$, we apply the regularity result of Proposition 2.2 of [17] to see that $\nabla\tilde{u}$ is bounded on the bounded subsets of $\bar{\tilde{\Omega}}_\lambda$ which contain neighbourhoods of 0 on $\partial\tilde{\Omega}_\lambda$. Hence, in the original variables, $\|\nabla u\|_\infty \leqslant K_0 \lambda^{1/p}$ on the subsets of $\bar{\Omega}$ which contain neighbourhoods of x_0 on $\partial\Omega$. The required estimate for u near $\partial\Omega$ now follows since $\partial\Omega$ is compact. This completes the proof. ∎

4. PROOF OF THEOREM A

We first show the existence of a positive solution $\zeta < u_\lambda < \rho_2$ of (1.1) for each ζ satisfying the conditions in Theorem A. From the definition of ζ, there exist $\xi \in (\rho_1, \rho_2)$ and a ball $B(x_0, r) \subset \Omega$ such that $\zeta > \xi$ in $B(x_0, r)$ and $\zeta(x_0) = \max_\Omega \zeta(x) \in (\rho_1, \rho_2)$. Let $w(\lambda, x_0)$ be as in (2.4). We know that $w(\lambda, x_0)$ is a subsolution of Eq. $(1.1)_\varepsilon$ for $\lambda > \mu\alpha^p \operatorname{dist}(x_0, \partial\Omega)^{-p}$. Therefore, it follows from the sub- and supersolution argument, Lemma 3.1, and Remark 3.2 that for $\lambda > \lambda^{**}_{x_0}$ (with x^* replaced by x_0) there exists a maximal positive solution u_λ of Eq. (1.1) in $[w(\lambda, x_0), \rho_2]$ such that

$$u_\lambda(x) > \min(c\lambda^{1/p} \operatorname{dist}(x, \partial\Omega), \tau) \qquad \text{for} \quad x \in \Omega.$$

Hence there exists $\lambda_1(\zeta) > \lambda^{**}_{x0}$ such that

$$\left|\frac{\partial\zeta}{\partial\nu}\right| < \left|\frac{\partial u_\lambda}{\partial\nu}\right| \qquad \text{on } \partial\Omega,$$

and thus,

$$u_\lambda(x) > \zeta(x) \qquad \text{for} \quad x \in \Omega \backslash \Omega_{\alpha(\mu/\lambda)^{1/p}}.$$

Since $\tau > \zeta(x_0)$, we see that for $\lambda > \lambda_1(\zeta)$, $u_\lambda(x) > \zeta(x)$ in Ω. This implies existence.

Now we show that for any $\tilde{\varepsilon} > 0$ we can choose $\lambda(\zeta, \tilde{\varepsilon})$ such that when $\lambda > \lambda(\zeta, \tilde{\varepsilon})$, any solution $u \in (\zeta, \rho_2]$ (we omit the subscript λ) of Eq. (1.1) has the behaviour of Proposition 3.6; that is, (3.5) holds.

First note that $\zeta > \xi$ in $B(x_0, r)$. Let $\sigma, \varepsilon > 0$ (ε sufficiently small) be such that $f(s) - \varepsilon > \sigma(s-\xi)^{p-1}$ for $s \in [\xi, \tau]$, where τ is as in Lemma 3.1. For

$$\lambda > \lambda_2(\zeta) := ((v_1/\sigma)^{1/p} + \mu^{1/p}\alpha)^p \, r^{-p},$$

with μ defined in Lemma 2.3 and α defined in Lemma 3.1, Lemma 3.3 shows that for any $u \in (\zeta, \rho_2]$,

$$u(x) > \tau \qquad \text{for} \quad x \in B(x_0, \alpha(\mu/\lambda)^{1/p}) \subset B(x_0, r - (v_1/\sigma)^{1/p} \, \lambda^{-1/p}).$$

Observe that $w(\lambda, x_0) < u$ in Ω for $\lambda > \lambda_2(\zeta)$. In fact, we know that $w(\lambda, x_0) \leqslant \tau$ in $B(x_0, \alpha(\mu/\lambda)^{1/p})$ and $w(\lambda, x_0) \leqslant 0$ for $\Omega \setminus B(x_0, \alpha(\mu/\lambda)^{1/p})$. The proof of Lemma 3.1 implies that the conclusion of Lemma 3.1 holds for u if $\lambda > \lambda_3(\zeta) = \max\{\lambda_2(\zeta), \mu\alpha^p\eta_0^{-p}\}$. The proof of Proposition 3.6 implies that for any $\tilde{\varepsilon} > 0$, there exists $\lambda(\zeta, \tilde{\varepsilon}) > \lambda_3(\zeta)$, such that for $\lambda > \lambda(\zeta, \tilde{\varepsilon})$, the behaviour of Proposition 3.6 holds for u.

We now prove uniqueness of u_λ. We shall assume that $p \neq 2$, the uniqueness result for $p = 2$ is known from Theorem 2 of [7].

Suppose there are sequences $\{\lambda_n\}$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\{u_{\lambda_n}\} \equiv \{u_n\}$, $\{u^*_{\lambda_n}\} \equiv \{u^*_n\}$ which are solutions of (1.1) with $\lambda = \lambda_n$ and with $u_n \not\equiv u^*_n$ in Ω and $u_n, u^*_n \in (\zeta, \rho_2]$. (The maximum principle implies that $\max_\Omega u_n < \rho_2$ and $\max_\Omega u^*_n < \rho_2$.) Without loss of generality, we assume that u_n is the maximal solution; thus, $u^*_n \leqslant u_n$. Let $w_n = (u_n - u^*_n)/\|u_n - u^*_n\|_\infty$. Then, $w_n \geqslant 0$ in Ω and $\|w_n\|_\infty = 1$. Therefore, w_n satisfies the problem

$$-L(w_n) := -\left[a_n^{ij} \frac{\partial}{\partial x_j} w_n \right]_{x_i} = \lambda_n f'(\xi_n) \, w_n \quad \text{in } \Omega, \qquad w_n = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

where $a_n^{ij}(x) = \int_0^1 (\partial a^i/\partial q_j) \, [sDu_n + (1-s) \, Du^*_n] \, ds$ and $a^i(q) = |q|^{p-2} \, q_i$ ($i = 1, 2, ..., N$) for $q = (q_1, q_2, ..., q_N) \in \mathbb{R}^N$, $\xi_n \in [u^*_n, u_n]$, L is a degenerate elliptic operator, and

$$\int_\Omega \sum_{ij} a_n^{ij}(x) \frac{\partial w_n}{\partial x_i} \frac{\partial w_n}{\partial x_j} \, dx \geqslant 0.$$

Now we show that if $\eta_n \in \Omega$ is such that $w_n(\eta_n) = 1$, then

$$\operatorname{dist}(\eta_n, \partial\Omega) \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.2}$$

In the contrary case, there exists a compact set $K \subset\subset \Omega$ such that $\eta_n \in K$ for all n large (choose a subsequence if necessary). Since $u_n \to \rho_2$ and $u_n^* \to \rho_2$ in K (by Proposition 3.6), we have $f'(\xi_n) < 0$ in K as $n \to \infty$ (by (F_3)). Also, for n sufficiently large, we can find a small neighbourhood D of K in Ω such that $f'(\xi_n) < 0$ on $\bar{D}$ and $\max_{\partial D} w_n < 1$. Let $\tilde{w}_n(x) = (w_n(x) - \max_{\partial D} w_n)^+$. We have $\tilde{w}_n \in W_0^{1,q}(D) \cap C^0(\bar{D})$ (for any $1 < q \leqslant \infty$) and there exists $A_n \subset D$ with $\operatorname{meas}(A_n) \neq 0$ and $\tilde{w}_n(x) > 0$ for $x \in A_n$. Multiplying both sides of Eq. (4.1) by $\tilde{w}_n$ and integrating on D, we derive a contradiction.

Now we use a blow-up argument as in [7] to deduce that (4.2) can not occur and thus we have uniqueness. Let $I_n = \{x \in \Omega : w_n(x) = 1\}$. I_n are closed nonempty sets. Let $\eta_n \in I_n$ be such that

$$\operatorname{dist}(\eta_n, \partial\Omega) = \operatorname{dist}(I_n, \partial\Omega),$$

and let $\tilde{\eta}_n$ be the point of $\partial\Omega$ closest to η_n. Suppose $\tilde{\eta}_n \to \tilde{\eta} \in \partial\Omega$ (for a subsequence). Choose coordinates such that $T_{\tilde{\eta}}(\partial\Omega) = \{x \in \mathbb{R}^N : x_1 = 0\}$ and $n_{s(\tilde{\eta})} = n_{\tilde{\eta}} = e_1 = (1, 0, ..., 0)$. By choosing subsequences if necessary, there are two cases to be considered:

(i) $\lambda_n^{1/p} d(\eta_n, \tilde{\eta}_n) \to \infty$ as $n \to \infty$,

(ii) $\lambda_n^{1/p} d(\eta_n, \tilde{\eta}_n) \leqslant Z$, $0 < Z < \infty$ for n sufficiently large.

In case (i), we have from Proposition 3.6 that $u_n(\eta_n) \geqslant \rho_2 - \delta/4$, $u_n^*(\eta_n) \geqslant \rho_2 - \delta/4$ when n is sufficiently large, where δ is as in (F_3). This and Proposition 3.6 imply that we can choose open sets $\Omega_n \subset\subset \Omega$ such that $I_n \subset \Omega_n$, $\max_{\partial\Omega_n} w_n < 1$ and

$$\rho_2 - \delta/2 \leqslant u_n(x) < \rho_2, \quad \rho_2 - \delta/2 \leqslant u_n^*(x) < \rho_2 \qquad \text{for} \quad x \in \Omega_n \tag{4.3}$$

and n sufficiently large. Since $f'(s) < 0$ for $s \in (\rho_2 - \delta/2, \rho_2)$, we derive a contradiction by the arguments similar to those in the proof of (4.2).

For case (ii), we change variables setting $X^n = \lambda_n^{1/p}(x - \tilde{\eta}_n)$. Note that this change depends on n. Let $\tilde{u}_n(X^n) = u_n(x)$, $\tilde{u}_n^*(X^n) = u_n^*(x)$, $\tilde{w}_n(X^n) = w_n(x)$, and $\tilde{\xi}_n(X^n) = \xi_n(x)$. We have that $\tilde{w}_n$ satisfies the problem

$$-\hat{L}_n(\tilde{w}_n) := -\left[\tilde{a}_n^{ij} \frac{\partial \tilde{w}_n}{\partial X_j^n}\right]_{X_i^n} = f'(\tilde{\xi}_n)\, \tilde{w}_n, \qquad \tilde{w}_n = 0 \quad \text{on } \partial\tilde{\Omega}_n,$$

where

$$\tilde{\Omega}_n \equiv \{X^n = \lambda_n^{1/p}(x - \tilde{\eta}_n) : x \in \Omega\},$$

$$\tilde{a}_n^{ij}(X^n) = \int_0^1 \frac{\partial a^i}{\partial q_j} [\xi D_{X^n} \tilde{u}_n + (1-\xi)\, D_{X^n} \tilde{u}_n^*]\, d\xi$$

and q, $a^i(q)$ are as above. Note that, in the new coordinates, $\tilde{w}_n(Z_n)=1$, where $Z_n=\lambda_n^{1/p}(\eta_n-\tilde{\eta}_n)$ is at distance at most Z from 0. Using Proposition 3.6 and a geometric argument similar to that in the proof of Theorem 2 of [7], we now show that $\tilde{u}_n\to z_0$, $\tilde{u}_n^*\to z_0$ in $C^1_{\mathrm{loc}}(T_1)$ as $n\to\infty$, where $T_1=\{x\in\mathbb{R}^N: x_1>0\}$ and z_0 is the unique positive solution of (3.4). In fact, if $\tilde{q}\in \operatorname{int} T_1$, then, for large n, $x^n(\tilde{q}):=\tilde{\eta}_n+\lambda_n^{-1/p}\tilde{q}\in\Omega$ and is close to $\partial\Omega$. By elementary geometry, $x^n(\tilde{q})=s(x^n(\tilde{q}))+t_n n_{s(x^n(\tilde{q}))}$, where $s(x^n(\tilde{q}))\in\partial\Omega$ is near $\tilde{\eta}$, $t_n=\lambda_n^{-1/p}(\tilde{q}_1+o(1))$, and $\tilde{q}_1$ is the first component of $\tilde{q}$. Recall that $\tilde{\eta}_n$ is near $\tilde{\eta}$ and $n_{\tilde{\eta}}=e_1$. By Proposition 3.6 and the definition of $\tilde{\eta}_n$, we have $u_n(\tilde{\eta}_n+\lambda_n^{-1/p}\tilde{q})=z_0(\tilde{q}_1)+o(1)$, and this holds locally uniformly in $\tilde{q}$ on $\operatorname{int}T_1$. This implies $\tilde{u}_n(X^n)\to z_0$ in $C^0_{\mathrm{loc}}(T_1)$ as $n\to\infty$ and similarly for $\tilde{u}_n^*$. The equations satisfied by $\tilde{u}_n$ and $\tilde{u}_n^*$ and the regularity of the p-Laplacian (see [17]) imply that $\tilde{u}_n\to z_0$, $\tilde{u}_n^*\to z_0$ in $C^1_{\mathrm{loc}}(T_1)$ as $n\to\infty$. Moreover, we claim that $\tilde{w}_n$ converges in $C^1_{\mathrm{loc}}(T_1)$ to a nontrivial non-negative bounded solution $\tilde{w}$ of

$$-\hat{L}(\tilde{w}):=-\left[\hat{a}^{ij}(x)\frac{\partial\tilde{w}}{\partial x_j}\right]_{x_i}=f'(z_0(x_1))\,\tilde{w}\quad\text{in } T_1,\qquad \tilde{w}=0\quad\text{on }\partial T_1. \tag{4.4}$$

In fact, it follows easily that $\tilde{a}_n^{ij}\to\hat{a}^{ij}$ in $C^0_{\mathrm{loc}}(T_1)$ as $n\to\infty$, where $\hat{a}^{ij}(x)=\partial a^i/\partial q_j\,(z_0'(x_1),0,\ldots,0)$ and thus, $\hat{a}^{ij}=0$ if $i\neq j$ and $\hat{a}^{11}=(p-1)\,|z_0'|^{p-2}$, $\hat{a}^{ii}=|z_0'|^{p-2}$ for $i\neq 1$. Since $z_0'(x_1)>0$ for all $0\leqslant x_1<\infty$, we have that $\hat{L}$ is uniformly elliptic on any compact subset of T_1 and so is $\hat{L}_n$ for n sufficiently large. Thus our claim can be obtained from the regularity of uniformly elliptic operators and a blow-up argument similar to that in the proof of Theorem 1.1 of [15] or that in the proof of Theorem 2 of [7]. Here $\tilde{w}$ is nontrivial because $\tilde{w}_n(Z_n)=1$ and $d(0,Z_n)\leqslant Z$. It is easily shown that $f'(\xi_n(x))\to f'(z_0(x_1))$.

Now we show that $\tilde{w}$ does not exist. The proof is divided into three steps. These steps are closely related to those in the proof of Proposition 2 in [7].

Step 1. We first find a solution $\hat{u}$ of

$$-(p-1)(|z_0'|^{p-2}\,u')'=f'(z_0)\,u \tag{4.5}$$

which is positive on $[0,\infty)$ and is not bounded as $x_1\to\infty$.

By differentiating the equation satisfied by z_0 with respect to x_1, we see that z_0' is a solution of Eq. (4.5). We know that $z_0'(x_1)>0$ for $x_1\in[0,\infty)$ and $z_0'(x_1)\to 0$ as $x_1\to\infty$. Let w denote the solution of Eq. (4.5) satisfying $w(0)=0$, $w'(0)=1$. Since $z_0'(x_1)>0$ on $[0,\infty)$, the Sturm comparison theorem implies that $w(x_1)>0$ for $x_1>0$. Hence, if we can show that $w(x_1)\to\infty$ as $x_1\to\infty$, we can define $\hat{u}=z_0'+w$ and this step will be proved. The fact that w is not bounded as $x_1 \to \infty$ can be obtained from the facts that

$$[(p-1)\,|z_0'|^{p-2}\,(z_0'w' - wz_0'')]' \equiv 0 \qquad \text{in } [0, \infty),$$
$$(p-1)|z_0'|^{p-2}\,(z_0'w' - wz_0'') \equiv (p-1)(z_0'(0))^{p-1} > 0$$

and $|z_0'(x_1)|^{p-2}\,z_0'(x_1) \to 0$, $|z_0'(x_1)|^{p-2}\,z_0'(x_1) \to 0$ as $x_1 \to \infty$.

Step 2. If Eq. (4.4) has a nontrivial bounded nonnegative solution v and $x_1 > 0$, then v can be chosen so that $T(x_1) \equiv \sup_{y \in \mathbb{R}^{N-1}} v(x_1, y)$ is achieved.

Obviously, there exist $y_n \in \mathbb{R}^{N-1}$ such that $v(x_1, y_n) \to T(x_1)$ as $n \to \infty$. Let $\tilde{v}_n(x_1, y) = v(x_1, y_n - y)$. It is easy to see that $\tilde{v}_n$ is a solution of Eq. (4.4) and that

$$\tilde{v}_n(x_1, 0) \to T(x_1) = \sup_{y \in \mathbb{R}^{N-1}} \tilde{v}_n(x_1, y) \qquad \text{as} \quad n \to \infty.$$

We now use an argument similar to that in our blow-up constructions to choose a subsequence of $\tilde{v}_n$ converging on compact subsets of T_1 to a nonnegative bounded solution $\bar{v}$ of Eq. (4.4). Moreover, $\bar{v}(x_1, 0) = T(x_1)$ by our choice of $\tilde{v}_n$. Since it is easy to see that

$$\sup_{y \in \mathbb{R}^{N-1}} \bar{v}(x_1, y) \leqslant \sup_{y \in \mathbb{R}^{N-1}} \tilde{v}_n(x_1, y) = T(x_1),$$

we see that $\sup_{y \in \mathbb{R}^{N-1}} \bar{v}(x_1, y) = \bar{v}(x_1, 0)$. This proves Step 2. Note that our argument shows that $\sup\{\bar{v}(x_1, y)\colon y \in \mathbb{R}^{N-1}\} \leqslant T(x_1)$ for all $x_1 \geqslant 0$. This will be useful later.

Step 3. We show that $\tilde{w}$ does not exist. If $\tilde{w}$ exists, using the notation of Step 2, we consider $r(x) = \tilde{w}(x)/\hat{u}(x_1)$, where $\hat{u}$ is the function defined in Step 1. Applying standard elliptic estimates on balls of radius $1/2$ and half balls with centres at points where $x_1 = 0$ and of radius 1, we see that $\nabla\tilde{w}$ is bounded on T_1. (Recall that $\tilde{w}$ is bounded on T_1 and $z_0''(x_1) > 0$ for $x_1 \in [0, \infty)$; hence $\hat{L}$ is uniformly elliptic on any compact subset of T_1.) Thus $\tilde{w}$ is uniformly continuous on T_1 and hence $T(x_1)$ is continuous. By Step 1 and the boundedness of $\tilde{w}$, it follows that $\lim_{x_1 \to \infty} T(x_1)/\hat{u}(x_1) = 0$. Thus, since $T(0) = 0$, we can find $0 < \tilde{x}_1 < \bar{x}_1$ such that

$$\sup\{T(x_1)/\hat{u}(x_1) : 0 \leqslant x_1 \leqslant \bar{x}_1\} = T(\tilde{x}_1)/\hat{u}(\tilde{x}_1).$$

By Step 2, $\tilde{w}$ can be chosen so that $\tilde{w}(\tilde{x}_1, y)$ achieves its maximum on $\mathbb{R}^{N-1}$ at 0. (Our construction of the new $\tilde{w}$ may decrease $T(x_1)$ for $x_1 \neq \tilde{x}_1$ but the maximum will still be attained at $\tilde{x}_1$.) By our construction, $r(x)$ achieves its maximum on $\{(x_1, y)\colon 0 \leqslant x_1 \leqslant \bar{x}_1, y \in \mathbb{R}^{N-1}\}$ at the *interior point* $(\tilde{x}_1, 0)$. However, since $\hat{u}$ satisfies Eq. (4.5), a simple calculation shows that r satisfies an elliptic equation

$$(p-1)(|z_0'|^{p-2}\, r'_{x_1})'_{x_1} + 2(p-1)\,|z_0'|^{p-2}\,(\hat{u}'/\hat{u})\, r'_{x_1} + (p-1)\,|z_0'|^{p-2}\,\Delta_{N-1} r \geqslant 0,$$

where Δ_{N-1} denotes the Laplacian in the y variables. Hence, by applying the maximum principle on compact sets, we see that $r(x_1, y)$ is constant if $0 \leqslant x_1 \leqslant \bar{x}_1$, $y \in \mathbb{R}^{N-1}$. This is impossible since $r = 0$ when $x_1 = 0$. This completes the proof.

Remark 4.1. From the proof of Theorem A we see that if ζ_1, $\zeta_2 \in C_0^\infty(\Omega)$ with $\max \zeta_i \in (\rho_1, \rho_2)\,(i = 1, 2)$ and there exist $x_i \in \Omega$ and some $r > 0$ such that $\zeta_i > \xi > \rho_1$ in $B(x_i, r) \subset \Omega$, then we can choose $\lambda_0 := \lambda_0(\zeta_1) = \lambda_0(\zeta_2)$ such that for $\lambda > \lambda_0$, (1.1) has exactly one large positive solution $\zeta_1 < \bar{u}_\lambda < \rho_2$ and $\zeta_2 < \bar{u}_\lambda < \rho_2$ in Ω. (Note that the relation between $\lambda_0(\zeta)$ and ζ in the proof of Theorem A depends only on r. Then we can choose $\lambda(\zeta_1, \tilde{\varepsilon}) = \lambda(\zeta_2, \tilde{\varepsilon})$ so that $\lambda_0(\zeta_1) = \lambda_0(\zeta_2)$.)

Now we obtain the following proposition.

PROPOSITION 4.2. *Let f satisfy* (F_1)–(F_3). *Then there exists $\sigma^* > 0$ independent of λ (sufficiently large) such that if u_λ is a positive solution of* (1.1) *with* $\max_\Omega u_\lambda \in (\rho_2 - \sigma^*, \rho_2)$ *then* $u_\lambda \equiv \bar{u}_\lambda$.

Proof. Suppose there exists sequences $\lambda_n \to \infty$ and $\{u_{\lambda_n}\}$ such that $u_{\lambda_n}\,(\neq \bar{u}_{\lambda_n})$ is a positive solution of (1.1) for each $n = 1, 2, \ldots$ and $\max_\Omega u_{\lambda_n} \to \rho_2$ as $n \to \infty$.

Let $x_n \in \Omega$ be a point at which $\max_\Omega u_{\lambda_n}$ is attained. For convenience, we divide the proof into two cases.

Case 1. $\{x_n\}$ is bounded away from $\partial\Omega$.

We define the functions $U_n(x) = u_{\lambda_n}(\lambda_n^{-1/p} x + x_n)$ in $B(0, R_n)$, where $R_n = \lambda_n^{1/p} \operatorname{dist}(x_n, \partial\Omega)$. Since $R_n \to \infty$ as $n \to \infty$, U_n is well defined in $B(0, R)$ for any $R > 0$ if n is sufficiently large. By assumption $0 < U_n < \rho_2$, $U_n(0) = \max_\Omega u_{\lambda_n} \to \rho_2$ as $n \to \infty$ and U_n satisfies $\Delta_p U_n + f(U_n) = 0$ in $B(0, R)$ for all sufficiently large n. Note that $\{f(U_n)\}$ is bounded in the L^∞-norm; thus by the regularity of the p-Laplacian (see [17]) we obtain (by choosing a subsequence) that $U_n \to U$ in $C^1(\overline{B(0, R)})$ as $n \to \infty$ where U, with $0 \leqslant U \leqslant \rho_2$, satisfies $\Delta_p U + f(U) = 0$ in $B(0, R)$ and $U(0) = \rho_2$. Since

$$-\Delta_p(\rho_2 - U) + M(\rho_2 - U)^{p-1} = -f(U) + M(\rho_2 - U)^{p-1}, \tag{4.6}$$

it follows from (F_3) that the right hand side of (4.6) is nonnegative. The strong maximum principle in [44] implies that $U \equiv \rho_2$ in $\overline{B(0, R)}$.

On the other hand, if n is sufficiently large, $\bar{u}_{\lambda_n}$ is the unique solution in the order interval $[w(\lambda_n, x_n), \rho_2]$. (This can be seen from the proof of Theorem A. In fact, since $\lambda_n^{1/p} \operatorname{dist}(x_n, \partial\Omega) \to \infty$ and $U_n \to \rho_2$ in $B(0, 2\alpha\mu^{1/p})$, then $\bar{u}_{\lambda_n} > \tau$ in $B(x_n, \alpha(\mu/\lambda_n)^{1/p})$, where $\tau = v_{\mu,\varepsilon}(0) < \rho_2$, and μ, α, and $v_{\mu,\varepsilon}$ are defined in Corollary 2.4. This implies that $\bar{u}_{\lambda_n} > w(\lambda_n, x_n)$ for n sufficiently large.) Thus, $u_{\lambda_n}(x) < w(\lambda_n, x_n)(x) < \tau$ at some point $x \in B(x_n, \alpha(\mu/\lambda_n)^{1/p})$. But this implies that $U_n(x) < \tau$ at some point $x \in B(0, \alpha\mu^{1/p})$ and therefore, $\{U_n\}$ cannot possess a subsequence which converges to ρ_2 uniformly in $\overline{B(0, \alpha\mu^{1/p})}$. This yields a contradiction and completes the proof of Case 1.

Case 2. $x_n \to \bar{x} \in \partial\Omega$ as $n \to \infty$.

To make use of the same argument as in Case 1, it suffices to show that

$$\lambda_n^{1/p} \operatorname{dist}(x_n, \partial\Omega) \to \infty \qquad \text{as} \quad n \to \infty$$

(for a subsequence). If not, we have that there exists $0 < Z < \infty$ such that

$$\lambda_n^{1/p} \operatorname{dist}(x_n, \partial\Omega) \leqslant Z \tag{4.7}$$

(choosing a subsequence if necessary). Define $\tilde{u}_{\lambda_n}(x) = u(\lambda_n^{-1/p} x + \tilde{x}_n)$ with $\tilde{x}_n = s(x_n) \in \partial\Omega$. By a blow-up argument similar to that in the proof of Theorem A, we have that there exists $\tilde{U}$ such that $\tilde{u}_{\lambda_n} \to \tilde{U}$ in $C^1_{\text{loc}}(T_1)$ as $n \to \infty$ and $\tilde{U}$ satisfies the problem

$$\Delta_p \tilde{U} + f(\tilde{U}) = 0 \text{ in } T_1, \qquad \tilde{U} = 0 \quad \text{on } \partial T_1, \tag{4.8}$$

where $T_1 = \{x \colon x_1 > 0\}$, as in the proof of Theorem A. By our assumption, there exists $x_0 \in T_1$ with $\operatorname{dist}(x_0, \partial T_1) \leqslant Z$ such that $\tilde{U}(x_0) = \rho_2$. Now arguments similar to those in Case 1 imply that $\tilde{U} \equiv \rho_2$ in $\bar{T}_1$. This contradiction implies $\lambda_n^{1/p} \operatorname{dist}(x_n, \partial\Omega) \to \infty$ as $n \to \infty$. ∎

5. EXISTENCE OF $\underline{u}_\lambda$

In this section we will use the mountain pass lemma to obtain the solution mentioned in Theorem B. For convenience, we change (1.1) to the singularly perturbed form. Let $\varepsilon = 1/\lambda$. Then (1.1) becomes

$$-\varepsilon \Delta_p u = f(u) \text{ in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

Denote the positive solution $\bar{u}_\lambda$ obtained in Theorem A by $\bar{u}_\varepsilon$ so that $\bar{u}_\varepsilon$ is a positive solution of Eq. (5.1).

Define $J_\varepsilon \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$J_\varepsilon(u) = \frac{\varepsilon}{p} \int_\Omega |Du|^p - \int_\Omega F(u),$$

where $F(u) = \int_0^u f(s)\,ds$. We first show that $\bar{u}_\varepsilon$ is a global minimizer of $J_\varepsilon(u)$. Since we only consider positive solutions u_ε of Eq. (5.1) with $\|u_\varepsilon\|_\infty \leqslant \rho_2$, we assume that $f \equiv 0$ for $s < 0$ and $s > \rho_2$.

LEMMA 5.1. *For $\varepsilon > 0$ sufficiently small, $\bar{u}_\varepsilon$ is the global minimizer of $J_\varepsilon(u)$. Moreover, $J_\varepsilon(\bar{u}_\varepsilon) < 0$.*

Proof. Since f is bounded in $[0, \rho_2]$, J_ε is sequentially weakly lower semicontinuous and coercive on $W_0^{1,p}(\Omega)$ and so J_ε possesses a global minimizer, which we denote by u_ε. From the regularity of the p-Laplacian (see [17]), $u_\varepsilon \in C_0^1(\Omega)$. We claim that $u_\varepsilon = \bar{u}_\varepsilon$ for ε sufficiently small. If not, there exists a sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$ as $m \to \infty$ and $u_{\varepsilon_m} \not\equiv \bar{u}_{\varepsilon_m}$. Then by Proposition 4.2, there exists $\sigma^* > 0$ (small) such that $\max_\Omega u_{\varepsilon_m} < \rho_2 - \sigma^*$ for $m = 1, 2, \ldots$.

For $\sigma > 0$, define $\Omega^\sigma = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \sigma\}$ and choose $a > 0$ such that $|\Omega^a| \int_{\rho_1}^{\rho_2} f(s)\,ds < |\Omega| \int_{\rho_2 - \sigma^*}^{\rho_2} f(s)\,ds$. This is possible since $\int_{\rho_2-\sigma^*}^{\rho_2} f(s)\,ds > 0$ and $|\Omega^a| \to 0$ as $a \to 0$. Next we choose $w \in C_0^\infty(\Omega)$ such that $w = \rho_2$ in $\Omega \setminus \Omega^a$. Then we have

$$\begin{aligned}
&J_{\varepsilon_m}(w) - J_{\varepsilon_m}(u_{\varepsilon_m}) \\
&\quad = \frac{\varepsilon_m}{p}\left(\int_\Omega |Dw|^p\,dx - \int_\Omega |Du_{\varepsilon_m}|^p\,dx\right) - \left(\int_\Omega F(w)\,dx - \int_\Omega F(u_{\varepsilon_m})\,dx\right) \\
&\quad = \frac{\varepsilon_m}{p}\int_\Omega |Du_{\varepsilon_m}|^p\,dx - \left(\int_{\Omega\setminus\Omega^a} F(\rho_2) + \int_{\Omega^a} F(w)\,dx - \int_\Omega F(u_{\varepsilon_m})\,dx\right) \\
&\quad = \frac{\varepsilon_m}{p}\int_\Omega |Dw|^p\,dx - \left[\int_\Omega (F(\rho_2) - F(u_{\varepsilon_m}))\,dx + \int_{\Omega^a} (F(w) - F(\rho_2)\,dx\right] \\
&\quad \leqslant \frac{\varepsilon_m}{p}\int_\Omega |Dw|^p\,dx - |\Omega| \int_{\rho_2-\sigma^*}^{\rho_2} f(s)\,ds + |\Omega^a| \int_{\rho_1}^{\rho_2} f(s)\,ds < 0
\end{aligned}$$

for m sufficiently large. Thus $J_{\varepsilon_m}(w) < J_{\varepsilon_m}(u_{\varepsilon_m})$, contradicting the fact that u_{ε_m} is a global minimizer.

Now we claim that

$$J_\varepsilon(\bar{u}_\varepsilon) \to -F(\rho_2)\,|\Omega| < 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Since $f(\rho_2) = 0$, for every $\hat{\varepsilon} > 0$ sufficiently small, there is $\hat{\delta} > 0$ such that for $s \in (\rho_2 - \hat{\delta}, \rho_2)$, $f(s) < \hat{\varepsilon}$. Also, from Lemma 3.4, there is $\hat{c}(\hat{\delta}) > 0$ such that for ε sufficiently small,

$$\bar{u}_\varepsilon > \rho_2 - \hat{\delta} \qquad \text{in} \quad \Omega \setminus \Omega^{\hat{c}(\hat{\delta})\, \varepsilon^{1/p}}.$$

We easily see from a blow-up argument that

$$\max_{\Omega^{\hat{c}(\hat{\delta})\, \varepsilon^{1/p}}} \varepsilon^{1/p} \, |D\bar{u}_\varepsilon| \leqslant C$$

for any $C > \max_{x_1 \in [0, \infty)} z_0''(x_1)$. In fact, we will show that

$$\max_{\Omega^{\hat{c}\varepsilon^{1/p}}} \varepsilon^{1/p} \, |D\bar{u}_\varepsilon| \leqslant C$$

for $C = \max_{x_1 \in [0, \infty)} z_0'(x_1) + 1$ (for example) for every $\hat{c} > \hat{c}(\hat{\delta})$. If not, there exist $\hat{c} > \hat{c}(\hat{\delta})$ and sequences $\{\varepsilon_n\}$ and $\{\bar{u}_{\varepsilon_n}\}$ such that, if

$$\varepsilon_n^{1/p} \, |D\bar{u}_{\varepsilon_n}(x_n)| = \max_{\Omega^{\hat{c}\varepsilon_n^{1/p}}} \varepsilon_n^{1/p} \, |D\bar{u}_{\varepsilon_n}(x)|,$$

then $\varepsilon_n^{1/p} \, |D\bar{u}_{\varepsilon_n}(x_n)| > C$. Define $X^n = \varepsilon_n^{-1/p}(x - \tilde{x}_n)$ and $\tilde{u}_n(X^n) = \bar{u}_{\varepsilon_n}(x)$, where $\tilde{x}_n = s(x_n) \in \partial\Omega$. We know from the proof of Theorem A that $\tilde{u}_n \to z_0$ in $C^1_{\mathrm{loc}}(T_{\hat{c}})$ as $n \to \infty$ (we can choose subsequences if necessary), where $T_{\hat{c}} = \{x \in T_1;\ 0 < x_1 < \hat{c}\}$. Since $\varepsilon_n^{1/p} \, |D_x \bar{u}_{\varepsilon_n}| = |D_{X^n} \tilde{u}_n|$, we have that

$$\varepsilon_n^{1/p} \, |D\bar{u}_{\varepsilon_n}| \to z_0'(x_1) \qquad \text{in} \quad C^0_{\mathrm{loc}}(T_{\hat{c}}) \quad \text{as } n \to \infty.$$

This also implies

$$\max_{\Omega^{\hat{c}\varepsilon_n^{1/p}}} \varepsilon_n^{1/p} \, |D\bar{u}_n| \leqslant C.$$

This contradicts our assumption above.

Multiplying both sides of Eq. (5.1) by $\bar{u}_\varepsilon$ and integrating on $E_\varepsilon := \Omega \setminus \Omega^{\hat{c}(\hat{\delta})\, \varepsilon^{1/p}}$, we have

$$\begin{aligned} \varepsilon \int_{E_\varepsilon} |D\bar{u}_\varepsilon|^p &= \int_{E_\varepsilon} f(\bar{u}_\varepsilon)\, \bar{u}_\varepsilon + \varepsilon \int_{\partial\Omega^{\hat{c}(\hat{\delta})\, \varepsilon^{1/p}} \setminus \partial\Omega} \bar{u}_\varepsilon \, |D\bar{u}_\varepsilon|^{p-2} \, (D\bar{u}_\varepsilon, \nu) \\ &< \hat{\varepsilon}\rho_2 \, |\Omega| + \rho_2 \varepsilon^{1/p} C. \end{aligned}$$

Moreover,

$$\varepsilon \int_{\Omega^{\hat{c}(\hat{\delta})\, \varepsilon^{1/p}}} |D\bar{u}_\varepsilon|^p \leqslant C \, |\Omega^{\hat{c}(\hat{\delta})\, \varepsilon^{1/p}}| = o(\varepsilon^{1/p}).$$

This implies that

$$\varepsilon \int_\Omega |D\bar{u}_\varepsilon|^p = \left(\int_{E_\varepsilon} + \int_{\Omega^{\hat{\varepsilon}(\hat{\delta})\,\varepsilon^{1/p}}} \right) |D\bar{u}_\varepsilon|^p \leqslant \hat{\varepsilon}\rho_2\, |\Omega| + O(\varepsilon^{1/p}).$$

Thus,

$$\varepsilon \int_\Omega |D\bar{u}_\varepsilon|^p \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

From Proposition 3.6 we have

$$\int_\Omega F(\bar{u}_\varepsilon) \to F(\rho_2)\, |\Omega| \qquad \text{as} \quad \varepsilon \to 0,$$

as claimed. ∎

THEOREM 5.2. *Assume that* $N \geqslant 3$, $p > 2$, *and* f *satisfies* (F_1)–(F_3). *Then there exists a positive solution* $\underline{u}_\varepsilon$ *of* (5.1) *with* $\underline{u}_\varepsilon \not\equiv \bar{u}_\varepsilon$ *and* $\|\underline{u}_\varepsilon\|_\infty > \rho_1$.

Proof. Let $E = W_0^{1,p}(\Omega)$. For $\rho \in [0, 1/2)$, we consider the functional $J_{\varepsilon,\rho}\colon E \to \mathbb{R}$ with

$$J_{\varepsilon,\rho}(u) = \frac{\varepsilon}{2} \int_\Omega \rho\, |Du|^2 + J_\varepsilon(u).$$

Clearly, for any ε and ρ, $J_{\varepsilon,\rho}$ is of class C^1 and $J_{\varepsilon,\rho}(0) = 0$.

We prove that for any $\varepsilon > 0$, $0 < \rho < 1/2$, $J_{\varepsilon,\rho}(u)$ satisfies the Palais–Smale condition (P-S) in E [37]. In fact, let $\{u_m\} \subset E$ be a sequence such that

$$|J_{\varepsilon,\rho}(u_m)| \leqslant C \text{ and } J'_{\varepsilon,\rho}(u_m) \to 0 \qquad \text{as} \quad m \to \infty \text{ for some constant } C.$$

From the inequality

$$\begin{aligned} \frac{\varepsilon}{2} \int_\Omega \rho\, |Du_m|^2 + \frac{\varepsilon}{p} \int_\Omega |Du_m|^p\, dx &= J_{\varepsilon,\rho}(u_m) + \int_\Omega F(u_m)\, dx \\ &\leqslant |J_{\varepsilon,\rho}(u_m)| + F(\rho_2)\, |\Omega|, \end{aligned}$$

we immediately see that $\{u_m\}$ is bounded in E.

We now show that $\{u_m\}$ has a convergent subsequence in E. Since $f \equiv 0$ for $s \leqslant 0$ and $s \geqslant \rho_2$, $\{f(u_m)\}$ is bounded in $L^\infty(\Omega)$. By arguments similar to those in the proof of Proposition 2.2 of [17], we have that

$$A^{-1}_{\varepsilon, p, \rho}\colon L^\infty(\Omega) \to C^1_0(\Omega)$$

is compact, where $A^{-1}_{\varepsilon, p, \rho}$ is the inverse operator of $-\varepsilon \operatorname{div}((\rho + |D\cdot|^{p-2})\, D\cdot)$ under the Dirichlet boundary condition. (Notice that Proposition 2.2 of [17] deals with the regularity of $A^{-1}_{\varepsilon, p, 0}$ (i.e., $\rho = 0$), but the conclusions remain valid for $\rho \in [0, 1/2)$. Since in the proof of Proposition 2.2 of [17], the author used the regularity results from [28, 42, 43], one can easily check that the results are also true for $\rho \in [0, 1/2)$.) This implies that there exists a subsequence $\{f(u_{m_i})\}$ of $\{f(u_m)\}$ in $L^\infty(\Omega)$ such that $\{A^{-1}_{\varepsilon, p, \rho} f(u_{m_i})\}$ is a convergent sequence in E. Therefore, the fact that $J'_{\varepsilon, \rho}(u_{m_i}) \to 0$ implies $u_{m_i} - A^{-1}_{\varepsilon, p, \rho} f(u_{m_i}) \to 0$ in E. Thus, $\{u_{m_i}\}$ is a convergent sequence in E. This implies that $J_{\varepsilon, \rho}$ satisfies the (P-S) condition.

For $2 < p < N$, by the conditions that $\lim_{s \to 0^+} f(s)/s^{p-1} = -m < 0$, and $f(0) = 0$, we can choose constants $C > 0$ and $q = p^2/(N-p)$ such that

$$J_{\varepsilon, \rho}(u) \geqslant (\varepsilon/p) \int_\Omega |Du|^p\, dx - \int_\Omega F(u)\, dx \geqslant (\varepsilon/p)\, \|u\|^p - \int_\Omega C\, |u|^{q+p}\, dx. \tag{5.2}$$

Since E is continuously embedded in $L^{q+p}(\Omega)$, we have

$$J_{\varepsilon, \rho}(u) \geqslant (\varepsilon/p)\, \|u\|^p - \int_\Omega C\, |u|^{q+p}\, dx \geqslant (\varepsilon/p - C'\, \|u\|^q)\, \|u\|^p.$$

Taking $\eta = (\theta\varepsilon)^{1/q}$ where $0 < \theta < 1/(2pC')$ and $\alpha = (\varepsilon/p - C'\eta^q)\, \eta^p \geqslant C'_1 \varepsilon^{N/p}$, we obtain $J_{\varepsilon, \rho}(u) > 0$ if $0 < \|u\| < \eta$ and $J_{\varepsilon, \rho}(u) \geqslant \alpha$ if $\|u\| = \eta$.

For $p \geqslant N$, and for every $q > 0$, (5.2) still holds. Since, by the Sobolev embedding theorem, E is continuously embedded in $C^0(\bar{\Omega})$, we have that

$$\begin{aligned} J_{\varepsilon, \rho}(u) &\geqslant (\varepsilon/p)\, \|u\|^p - \int_\Omega C\, |u|^{q+p}\, dx \\ &\geqslant (\varepsilon/p)\, \|u\|^p - C'(\Omega)\, \|u\|^{q+p}_\infty \\ &\geqslant (\varepsilon/p - C''\, \|u\|^q)\, \|u\|^p. \end{aligned}$$

Choosing $\eta = (\theta\varepsilon)^{1/q}$ where $0 < \theta < 1/(2pC'')$ and $\alpha = (\varepsilon/p - C''\eta^q)\, \eta^p \geqslant C'_1 \varepsilon^{1+p/q}$ we obtain $J_{\varepsilon, \rho}(u) > 0$ if $0 < \|u\| < \eta$ and $J_{\varepsilon, \rho}(u) \geqslant \alpha$ if $\|u\| = \eta$. Note that α is independent of ρ in both cases.

Next we show that there is $e \in E \setminus \{u \in E : \|u\| \leqslant \eta\}$ such that $J_{\varepsilon,\rho}(e) \leqslant 0$. Let $\bar{u}_{\varepsilon_0} \equiv \bar{u}_{\lambda_0}$ be the solution of the problem

$$-\operatorname{div}(|Du|^{p-2}Du) = \lambda_0 f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

obtained in Theorem A with $\varepsilon_0 = 1/\lambda_0$. Then $\max \bar{u}_{\varepsilon_0}$ is near ρ_2 if ε_0 is sufficiently small. Moreover, for a fixed $\varepsilon_0 > 0$, it follows from Lemma 5.1 that $J_{\varepsilon_0}(\bar{u}_{\varepsilon_0}) < 0$. Since $\varepsilon_0 \int_\Omega |D\bar{u}_{\varepsilon_0}|^p \to 0$ as $\varepsilon_0 \to 0$, then $\varepsilon_0 \int_\Omega \rho\, |D\bar{u}_{\varepsilon_0}|^2 \to 0$ for any $\rho \in [0, 1/2)$ (note that $p > 2$). This implies that for every $\rho \in [0, 1/2)$, there is $\varepsilon_0 > 0$ such that $J_{\varepsilon_0,\rho}(\bar{u}_{\varepsilon_0}) < 0$.

Now define

$$U_\varepsilon(x) = \begin{cases} \bar{u}_{\varepsilon_0}((\varepsilon_0/\varepsilon)^{1/p} x), & \text{if } x \in (\varepsilon/\varepsilon_0)^{1/p}\,\Omega, \\ 0, & \text{if } x \in \Omega \setminus (\varepsilon/\varepsilon_0)^{1/p}\,\Omega. \end{cases}$$

Note that $0 \in \Omega$ and $(\varepsilon/\varepsilon_0)^{1/p}\,\Omega \subset \Omega$ if $\varepsilon < \varepsilon_0$. For $0 < \varepsilon < \varepsilon_0$, we have $U_\varepsilon \in E$. Then it follows that

$$\begin{aligned} J_{\varepsilon,\rho}(U_\varepsilon) &= \frac{\varepsilon}{2} \int_{(\varepsilon/\varepsilon_0)^{1/p}\,\Omega} \rho\, |D\bar{u}_{\varepsilon_0}((\varepsilon_0/\varepsilon)^{1/p} x)|^2\, dx + \frac{\varepsilon}{p} \int_{(\varepsilon/\varepsilon_0)^{1/p}\Omega} |D\bar{u}_{\varepsilon_0}((\varepsilon_0/\varepsilon)^{1/p} x)|^p\, dx \\ &\quad - \int_{(\varepsilon/\varepsilon_0)^{1/p}\,\Omega} F(\bar{u}_{\varepsilon_0}((\varepsilon_0/\varepsilon)^{1/p} x)\, dx \\ &= (\varepsilon/\varepsilon_0)^{N/p} \left[(\varepsilon/\varepsilon_0)(\varepsilon_0/\varepsilon)^{2/p}\,(\varepsilon_0/2) \int_\Omega \rho\, |D\bar{u}_{\varepsilon_0}|^2 + J_{\varepsilon_0}(\bar{u}_{\varepsilon_0}) \right] \\ &< (\varepsilon/\varepsilon_0)^{N/p}\, J_{\varepsilon_0,\rho}(\bar{u}_{\varepsilon_0}) < 0. \end{aligned}$$

This implies that U_ε satisfies our requirement. In fact, we have

$$\varepsilon\, \|U_\varepsilon\|^p = (\varepsilon/\varepsilon_0)^{N/p}\, \varepsilon_0\, \|\bar{u}_{\varepsilon_0}\|^p,$$

and so,

$$\|U_\varepsilon\| = (\varepsilon/\varepsilon_0)^{(N-p)/p^2}\, \|\bar{u}_{\varepsilon_0}\|.$$

Therefore, for $p < N$, we have $\|U_\varepsilon\| > \eta$ if we choose $0 < \theta < (1/\varepsilon_0)\, \|\bar{u}_{\varepsilon_0}\|^{p^2/(N-p)}$. For $p \geqslant N$, we easily see that $\|U_\varepsilon\| > \eta$ for $0 < \theta < 1/(2pC'')$.

Now we apply the mountain pass lemma [37] to the functional $J_{\varepsilon,\rho}$ and obtain a critical point $\underline{u}_{\varepsilon,\rho}$ with critical value $J_{\varepsilon,\rho}(\underline{u}_{\varepsilon,\rho}) = \inf_{g \in \tilde{\Gamma}} \max_{u \in g([0,1])} J_{\varepsilon,\rho}(u)$, where $\tilde{\Gamma} = \{g \in C([0,1]; E) : g(0) = 0,\ g(1) = U_\varepsilon\}$. Since f is bounded, by regularity of the p-Laplacian, (see [17]), we have that $\underline{u}_{\varepsilon,\rho} \in C^{1,\alpha}(\bar{\Omega})$. It is clear that for $0 < \varepsilon < \varepsilon_0$ (ε_0 is independent of ρ) and any $\rho \in [0, 1/2)$, $\underline{u}_{\varepsilon,\rho}$ is a nontrivial solution of the problem

$$-\varepsilon \operatorname{div}((\rho + |Du|^{p-2})\, Du) = f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{5.3}$$

Moreover, the maximum principle implies $0 < \underline{u}_{\varepsilon,\rho} \leqslant \rho_2$. Since

$$\varepsilon\rho \int_\Omega |D\underline{u}_{\varepsilon,\rho}|^2 + \varepsilon \int_\Omega |D\underline{u}_{\varepsilon,\rho}|^p = \int_\Omega f(\underline{u}_{\varepsilon,\rho})\, \underline{u}_{\varepsilon,\rho},$$

then

$$\|\underline{u}_{\varepsilon,\rho}\|_{W_0^{1,p}(\Omega)} \leqslant C,$$

where C depends on ε but is independent of ρ. By arguments similar to those in the proof of Proposition 2.2 of [17], we have $\|\underline{u}_{\varepsilon,\rho}\|_{C^{1,\beta}(\Omega)} \leqslant C$ for $0 < \beta < 1$ where C depends on ε but is independent of ρ. (Such results can be obtained directly from [28, 42, 43].) This implies that for $0 < \varepsilon < \varepsilon_0$,

$$\underline{u}_{\varepsilon,\rho} \to \underline{u}_\varepsilon \qquad \text{in } C^1(\bar{\Omega}) \quad \text{as } \rho \to 0.$$

Since $J_{\varepsilon,\rho}(\underline{u}_{\varepsilon,\rho}) \geqslant \alpha > 0$, α is independent of ρ, and

$$J_{\varepsilon,\rho}(\underline{u}_{\varepsilon,\rho}) \to J_\varepsilon(\underline{u}_\varepsilon) > \alpha \qquad \text{as} \quad \rho \to 0,$$

we know from Lemma 5.1 that $\underline{u}_\varepsilon \not\equiv \bar{u}_\varepsilon$. Moreover,

$$J_\varepsilon(\underline{u}_\varepsilon) = \inf_{g \in \tilde{\Gamma}} \max_{u \in g([0,1])} J_\varepsilon(u).$$

From the maximum principle we have $\max_{\bar{\Omega}} \underline{u}_\varepsilon > \rho_1$. ∎

Remark 5.3. Taking $\rho = 0$, we see that the conclusion of Theorem 5.2 holds also for $1 < p \leqslant 2$. We present a bad version of Theorem 5.2 here in order to obtain the asymptotic behaviour of $\underline{u}_\varepsilon$ as $\varepsilon \to 0$ below.

6. PROOF OF THEOREM B

To obtain the first claim of Theorem B, we define g: $[0, 1] \to E$ by $g(t) = tU_\varepsilon$. Then we have

$$
\begin{aligned}
J_\varepsilon(tU_\varepsilon) &= \frac{\varepsilon}{p} \int_{(\varepsilon/\varepsilon_0)^{1/p}\Omega} t^p \, |D\bar{u}_{\varepsilon_0}((\varepsilon_0/\varepsilon)^{1/p} x)|^p \, dx \\
&\quad - \int_{(\varepsilon/\varepsilon_0)^{1/p}\Omega} F(t\bar{u}_{\varepsilon_0}((\varepsilon_0/\varepsilon)^{1/p} x)) \, dx \\
&\leqslant (\varepsilon/\varepsilon_0)^{N/p} \left(\frac{\varepsilon_0}{p} \int_\Omega |D\bar{u}_{\varepsilon_0}(x)|^p \, dx - F(\rho_1) \, |\Omega| \right) \leqslant C_1 \varepsilon^{N/p},
\end{aligned}
$$

and thus $J_\varepsilon(\underline{u}_\varepsilon) \leqslant \max_{t \in [0,1]} J_\varepsilon(tU_\varepsilon) \leqslant C_1 \varepsilon^{N/p}$. Since $J_\varepsilon(\underline{u}_\varepsilon) \geqslant \alpha$, claims (i) and (ii) of Theorem B hold.

To show (iii) of Theorem B, we first define a class of perturbations $g_\theta(s)$ (with $\theta > 0$ small) of $f(s)$ such that

$$
\gamma(\theta) + f(s) \leqslant g_\theta(s) \leqslant f(s) + \Gamma(\theta) \qquad \text{for} \quad s \in [0, \rho_2 + 1) \tag{6.1}
$$

and g_θ satisfies (H_1)–(H_4) in Theorem A.5 of the Appendix for any $\theta > 0$ sufficiently small, where $\gamma(\theta), \Gamma(\theta) \in C^0([0, 1/8])$ satisfy $\gamma(0) = \Gamma(0) = 0$ and $\Gamma(\theta) \geqslant \gamma(\theta) > 0$ for $0 < \theta \leqslant 1/8$.

We denote the three zeros of g_θ by $\rho_0(\theta)$, $\rho_1(\theta)$, and $\rho_2(\theta)$. Then it is clear that

$$
0 < \rho_0(\theta) < \rho_1(\theta) < \rho_1 < \rho_2 < \rho_2(\theta) \tag{6.2}
$$

and

$$
\lim_{\theta \to 0} \rho_0(\theta) = 0, \qquad \lim_{\theta \to 0} |\rho_i(\theta) - \rho_i| = 0 \quad \text{for} \quad i = 1, 2, \tag{6.3}
$$

where ρ_1, ρ_2 are the positive zeros of f. Moreover, for any $\theta > 0$, there is a unique $\mu(\theta) \in (\rho_1(\theta), \rho_2(\theta))$ such that

$$
\int_{\rho_0(\theta)}^{\mu(\theta)} g_\theta(s) \, ds = 0.
$$

Note that the zeros of g_θ depend on θ; we assume that g_θ is chosen so that there exists $C>0$ independent of θ such that

$$|g_\theta(s)| \leqslant C\,|s-\rho_i(\theta)|^{p-1}$$

for s near $\rho_i(\theta)$ and $i=0, 1, 2$. Moreover, we assume that

$$\lim_{s\to\rho_0(\theta)^+} g_\theta(s)/(s-\rho_0(\theta))^{p-1} = -m_1(\theta) < 0$$

and $m_2 \leqslant m_1(\theta) \leqslant m_3$ for all θ small, where $0<m_2\leqslant m_3$ are independent of θ.

For example, if f is the example given in the Introduction, we may choose the perturbations $g_\theta(s)$ ($\theta>0$ sufficiently small) to be

$$g_\theta(s) = \begin{cases} -m\psi_\theta^1(s)\,\psi_\theta^2(s)\,\psi_\theta^3(s), & \text{for} \quad p\geqslant 2, \\ -m\,|(s-\theta)|^{p-2}\,(s-\theta)((a-\theta)-s)((1+\theta)-s), & \text{for} \quad 1<p<2, \end{cases}$$

where

$$\psi_\theta^1(s) = |s-\theta|^{p-2}\,(s-\theta),$$
$$\psi_\theta^2(s) = |(a-\theta)-s|^{p-2}\,((a-\theta)-s), \; 0<a<1/4,$$
$$\psi_\theta^3(s) = |(1+\theta)-s|^{p-2}\,((1+\theta)-s).$$

(Note that for $p>2$, $g_\theta \in C^1((0,\infty)) \cap C^0([0,\infty))$.)

We will use the following lemma.

LEMMA 6.1. *Let p, Ω be as in Theorem* B *and $\underline{u}_\varepsilon$ be the positive solution obtained in Theorem* 5.2. *Then there exists ε^* such that for $0<\varepsilon<\varepsilon^*$, $\underline{u}_\varepsilon$ satisfies*

$$\left|\frac{\partial \underline{u}_\varepsilon}{\partial \nu}(x)\right| \leqslant (C/\varepsilon)\, e^{-\sigma\varepsilon^{-1/p}} \qquad \textit{for all} \quad x\in\partial\Omega, \tag{6.4}$$

where C and σ are independent of ε and $\nu(x) = -n_x$ denotes the outward normal vector at $x\in\partial\Omega$.

Proof. We show that there exists $\tilde{\rho}>0$ independent of ε such that

$$l_{\tilde{\rho}} = \sup_{\Omega^{\tilde{\rho}}} \underline{u}_\varepsilon < \hat{\mu}, \tag{6.5}$$

where $\hat{\mu}\in(\rho_1,\rho_2)$ is such that $\int_0^{\hat{\mu}} f(s)\,ds = 0$.

By the moving plane method near $\partial\Omega$, as in the Appendix, we can find $\tilde{\sigma}>0$ independent of ε and ρ such that for any $y\in\Omega^{\tilde{\sigma}}=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)<\tilde{\sigma}\}$, there is a fixed-size cone $K_y\subset\Omega^{\tilde{\sigma}}$ with vertex at y and $\underline{u}_{\varepsilon,\rho}(y)=\min_{x\in K_y}\underline{u}_{\varepsilon,\rho}(x)$.

From the proof of Theorem 5.2, for $0<\varepsilon<\varepsilon_0$, $\underline{u}_{\varepsilon,\rho}\to\underline{u}_\varepsilon$ in $C^1(\bar{\Omega})$ as $\rho\to 0$ and $\underline{u}_\varepsilon$ has the same properties as $\underline{u}_{\varepsilon,\rho}$. Thus, for any $y\in\Omega^{\tilde{\sigma}}$, there is a fixed-size cone $K_y\subset\Omega^{\tilde{\sigma}}$ with vertex at y and $\underline{u}_\varepsilon(y)=\min_{x\in K_y}\underline{u}_\varepsilon(x)$. Let $\tilde{\delta}\in(0,\hat{\mu}-\rho_1)$. We claim that there exists $0<\varepsilon_1<\varepsilon_0$ such that for $0<\varepsilon<\varepsilon_1$, $\sup_{\Omega^{\tilde{\sigma}}}\underline{u}_\varepsilon<\rho_1+\tilde{\delta}$. To see this, fix $y\in\Omega^{\tilde{\sigma}}$ and choose a positive radially symmetric function $\phi\in C_0^\infty(\Omega)$ so that $\operatorname{supp}\phi\subset K_y$ and $\rho_1<\max\phi<\rho_1+\tilde{\delta}$. Then there exists $\tilde{x}_1\in K_y$ and $\tilde{r}>0$ such that $\phi\geqslant\xi>\rho_1$ in $B(\tilde{x}_1,\tilde{r})\subset K_y$. By Theorem A, there exists a constant $\varepsilon_1=\varepsilon_1(\phi)<\varepsilon_0$ such that, for $0<\varepsilon<\varepsilon_1$, $\bar{u}_\varepsilon$ is the unique solution between ϕ and ρ_2. Suppose that for $0<\varepsilon<\varepsilon_1$, $\underline{u}_\varepsilon(y')\geqslant\rho_1+\tilde{\delta}$ at some point $y'\in\Omega^{\tilde{\sigma}}$. Then $\underline{u}_\varepsilon\geqslant\rho_1+\tilde{\delta}$ in $K_{y'}$ and consequently $\underline{u}_\varepsilon>\phi'$ for some $\phi'\in C_0^\infty(\Omega)$, which is a translation of ϕ. Since K_y and $K_{y'}$ have the same fixed size, then there exists $\tilde{x}_2\in K_{y'}$ such that $B(\tilde{x}_2,\tilde{r})\subset K_{y'}$ and $\phi'\geqslant\xi>\rho_1$ in $B(\tilde{x}_2,\tilde{r})$ (since ϕ' is a translation of ϕ). Remark 4.1 then implies $\varepsilon_1(\phi)=\varepsilon_1(\phi')$. This gives $\underline{u}_\varepsilon\equiv\bar{u}_\varepsilon$, a contradiction. Let $\tilde{\rho}=\tilde{\sigma}$. Thus, (6.5) holds.

Now fix a point $x^*\in\partial\Omega$. Without loss of generality, we may assume that $x^*=0$ and $\nu(x^*)=e_1=(1,0,...,0)$, where $\nu(x)$ is the outward normal vector at $x\in\partial\Omega$.

To construct a family of supersolutions to Eq. (5.1), we consider the following ordinary differential equation with an arbitrary small $\theta>0$

$$\varepsilon\,(|w'|^{p-2}\,w')'+g_\theta(w)=0 \text{ in } (0,\gamma),\quad w'(0)=0,\qquad w(\gamma)=\rho_0(\theta),\tag{6.6}$$

where $0<\gamma<\tilde{\rho}$ is independent of ε and θ. From Theorem A.5 there exists $\bar{\varepsilon}(\theta)>0$ such that for $0<\varepsilon<\bar{\varepsilon}(\theta)$, (6.6) has a solution $w_{\theta,\varepsilon}$ with the following properties:

(1) $|w_{\theta,\varepsilon}-\rho_0(\theta)|\leqslant Ce^{-\sigma/\varepsilon^{1/p}}$ in any closed subinterval of $(0,\gamma]$, where $C>0$, $\sigma>0$ are independent of ε and θ,

(2) $w_{\theta,\varepsilon}(0)\to\mu(\theta)$ as $\varepsilon\to 0$.

(From the proof of Theorem A.5, σ in (1) depends on $m_1(\theta)$, but since $m_2\leqslant m_1(\theta)\leqslant m_3$, we can choose σ in (1) that does not depend on θ, but σ depends on m_2.)

We claim that we can choose $\bar{\varepsilon}>0$ independent of θ such that for $0<\varepsilon<\bar{\varepsilon}$, properties (1) and (2) of $w_{\theta,\varepsilon}$ still hold. In fact, we easily see that $w_{\theta,\varepsilon}-\rho_0(\theta)$ satisfies the problem (A.19) in Corollary A.7. Then if we choose $\bar{\varepsilon}$ as in Corollary A.7, we have from Corollary A.7 that if $0<\varepsilon<\bar{\varepsilon}$, then (1) and (2) hold. Choosing $\varepsilon^*=\min\{\varepsilon_1, \bar{\varepsilon}\}$, we have (6.5), and properties (1) and (2) hold for $0<\varepsilon<\varepsilon^*$ and any small $\theta>0$.

We now define $w_{\theta,\varepsilon,t}(x)=w_{\theta,\varepsilon}(x_1+t)$ for $0\leqslant t\leqslant\gamma$. Clearly $w_{\theta,\varepsilon,t}$ satisfies

$$\varepsilon\Delta_p u+g_\theta(u)=0 \qquad \text{in} \quad \Omega'_t=\Omega\cap\{x\in\mathbb{R}^N: x\cdot e_1=x_1>-t\}$$

and there exists $t_0>0$ (depending on θ and ε) such that $w_{\theta,\varepsilon,t_0}>\underline{u}_\varepsilon$ in Ω'_{t_0} (since $\mu(\theta)\to\hat{\mu}$ as $\theta\to 0$, we can choose θ so small that $\mu(\theta)>l_{\tilde{\rho}}$). By the sweeping out result as in Proposition 2.5, we have

$$w_{\theta,\varepsilon,\gamma}\geqslant\underline{u}_\varepsilon \qquad \text{in } \Omega'_\gamma. \tag{6.7}$$

In fact, by the convexity of Ω, $\Omega'_t=\Omega\cap\{x\in\mathbb{R}^N: -t<x_1<0\}$ and for any $t_0\leqslant t_1\leqslant\gamma$, $w_{\theta,\varepsilon,t_1}(x)|_{x_1=-t_1}=w_{\theta,\varepsilon}(0)=\mu(\theta)+o(1)>\underline{u}_\varepsilon(x)|_{x_1=-t_1}$ for ε sufficiently small, $w_{\theta,\varepsilon,t_1}|_{\partial\Omega}>\underline{u}_\varepsilon|_{\partial\Omega}$. Thus, arguments similar to those in the proof of Proposition 2.5 imply that

$$w_{\theta,\varepsilon,t_1}(x)\geqslant\underline{u}_\varepsilon(x) \qquad \text{for} \quad x\in\Omega'_{t_1} \quad \text{and} \quad t_1\in[t_0,\gamma].$$

This implies that (6.7) holds.

By compactness of $\partial\Omega$ and the properties of $w_{\theta,\varepsilon,\gamma}$, we obtain from (6.7) that

$$\underline{u}_\varepsilon\leqslant\rho_0(\theta)+Ce^{-\sigma/\varepsilon^{1/p}} \qquad \text{in} \quad \overline{\Omega^{\gamma/2}}. \tag{6.8}$$

Since C and σ are independent of θ, letting $\theta\to 0$, we have

$$\underline{u}_\varepsilon\leqslant Ce^{-\sigma/\varepsilon^{1/p}} \qquad \text{in} \quad \overline{\Omega^{\gamma/2}}. \tag{6.9}$$

(Note that $\rho_0(\theta)\to 0$ as $\theta\to 0$.) On the other hand, it follows from (6.7) that

$$w_{0,\varepsilon,\gamma}\geqslant\underline{u}_\varepsilon \qquad \text{in} \quad \Omega'_\gamma, \tag{6.10}$$

where $w_{0,\varepsilon}=w_\varepsilon$ is the solution given in Corollary A.7. (In fact, we can show that $w_{\theta,\varepsilon}\to w_\varepsilon$ in $C^1([0,\gamma])$ as $\theta\to 0$). Since $x^*\in\partial\Omega'_\gamma$, it follows from (6.10) that

$$\frac{\partial\underline{u}_\varepsilon}{\partial\nu}(x^*)\leqslant\frac{\partial w_{0,\varepsilon,\gamma}}{\partial\nu}(0)=w'_{0,\varepsilon}(\gamma). \tag{6.11}$$

Notice that $w_{0,\varepsilon}$ satisfies problem (A.19) of the Appendix. It follows from Corollary A.7 that

$$w'_{0,\varepsilon}(\gamma) \leqslant (C/\varepsilon)\, e^{-\sigma\varepsilon^{-1/p}},$$

where C and σ are independent of ε. This completes the proof. ∎

Remark 6.2. The proof of Lemma 6.1 also implies that the maximum of $\underline{u}_\varepsilon$ can not be attained near $\partial\Omega$. In fact, the proof of Lemma 6.1 shows that $\max_{\bar{\Omega}^{\gamma/2}} \underline{u}_\varepsilon < \rho_1$. A similar argument to that in the proof of Lemma 7.2 implies that if $x_0 \in \Omega$ is a local maximum point of $\underline{u}_\varepsilon$, then $\underline{u}_\varepsilon(x_0) > \rho_1$.

Proof of part (iii) *of Theorem* B. Pohozaev's identity for $\underline{u}_\varepsilon$ (see [16]),

$$(1-N/p)\,\varepsilon \int_\Omega |D\underline{u}_\varepsilon|^p\,dx + N\int_\Omega F(\underline{u}_\varepsilon)\,dx = \varepsilon(1-1/p)\int_{\partial\Omega} |D\underline{u}_\varepsilon|^p\,(x\cdot\nu)\,d\sigma,$$

and (i) and (ii) of Theorem B and Lemma 6.1 imply that

$$\begin{aligned}\varepsilon \int_\Omega |D\underline{u}_\varepsilon|^p\,dx &= NJ_\varepsilon(\underline{u}_\varepsilon) + \varepsilon(1-1/p)\int_{\partial\Omega} |D\underline{u}_\varepsilon|^p\,(x\cdot\nu)\,d\sigma \\ &< C\varepsilon^{N/p} + C\frac{\varepsilon}{p}\varepsilon^{N/p} < C\varepsilon^{N/p}.\end{aligned}$$

Next, for each $0<\sigma<\hat{\mu}-\rho_1$, we define $\Omega_{\varepsilon,\rho_1+\sigma} \equiv \{x \in \Omega : \underline{u}_\varepsilon(x) > \rho_1+\sigma\}$ and we shall prove that $|\Omega_{\varepsilon,\rho_1+\sigma}| < C\varepsilon^{N/p}$ for some $C>0$ independent of ε. Choose $k>0$ independent of ε such that $f(\underline{u}_\varepsilon) > k$ in $\Omega_{\varepsilon,\rho_1+\sigma/2}$; this is possible by assumptions (F_1)–(F_3) and Proposition 4.2, since we know from Proposition 4.2 that $\max_\Omega \underline{u}_\varepsilon < \rho_2 - \sigma_1$ with $\sigma_1>0$ independent of ε. Then we have

$$\begin{aligned}|\Omega_{\varepsilon,\rho_1+\sigma}| &< \frac{2}{\sigma}\int_{\Omega_{\varepsilon,\rho_1+\sigma}} (\underline{u}_\varepsilon - \rho_1 - \sigma/2)\,dx < \frac{2}{\sigma}\int_{\Omega_{\varepsilon,\rho_1+\sigma/2}} (\underline{u}_\varepsilon - \rho_1 - \sigma/2)\,dx \\ &\leqslant \frac{2}{k\sigma}\int_{\Omega_{\varepsilon,\rho_1+\sigma/2}} f(\underline{u}_\varepsilon)(\underline{u}_\varepsilon - \rho_1 - \sigma/2)\,dx \\ &= \frac{2\varepsilon}{k\sigma}\int_{\Omega_{\varepsilon,\rho_1+\sigma/2}} -(\underline{u}_\varepsilon - \rho_1 - \sigma/2)\,\mathrm{div}(|D\underline{u}_\varepsilon|^{p-2} D\underline{u}_\varepsilon)\,dx \\ &< \frac{2\varepsilon}{k\sigma}\int_{\Omega_{\varepsilon,\rho_1+\sigma/2}} |D\underline{u}_\varepsilon|^p\,dx < \frac{2\varepsilon}{k\sigma}\int_\Omega |D\underline{u}_\varepsilon|^p\,dx \leqslant \frac{2C}{k\sigma}\varepsilon^{N/p}.\end{aligned}$$

Noting that there is a constant $K>0$ such that $u^p \leqslant -KF(u)$ for $0 \leqslant u \leqslant \rho_1+\sigma$, we have

$$\begin{aligned}
\int_\Omega \underline{u}_\varepsilon^p\,dx &= \int_{\Omega\setminus\Omega_{\varepsilon,\rho_1+\sigma}} \underline{u}_\varepsilon^p\,dx + \int_{\Omega_{\varepsilon,\rho_1+\sigma}} \underline{u}_\varepsilon^p\,dx \\
&\leqslant -K\int_{\Omega\setminus\Omega_{\varepsilon,\rho_1+\sigma}} F(\underline{u}_\varepsilon)\,dx + \int_{\Omega_{\varepsilon,\rho_1+\sigma}} \underline{u}_\varepsilon^p\,dx \\
&\leqslant KJ_\varepsilon(\underline{u}_\varepsilon) + K\int_{\Omega_{\varepsilon,\rho_1+\sigma}} F(\underline{u}_\varepsilon)\,dx + \int_{\Omega_{\varepsilon,\rho_1+\sigma}} \underline{u}_\varepsilon^p\,dx \\
&\leqslant KC_1\varepsilon^{N/p} + (\rho_2^p + KF(\rho_2))\,|\Omega_{\varepsilon,\rho_1+\sigma}| \\
&\leqslant C\varepsilon^{N/p},
\end{aligned}$$

and our proof of part (iii) of Theorem B is completed.

Remark 6.3. We easily see that $\underline{u}_\varepsilon < \bar{u}_\varepsilon$ in Ω as $\varepsilon \to 0$. Thus, $\underline{u}_\varepsilon$ is a small solution.

7. PROOF OF THEOREM C

Let x_ε be a point at which $\underline{u}_\varepsilon$ attains a local maximum. We know that $\{x_\varepsilon\}$ is bounded away from $\partial\Omega$. Let $U_\varepsilon(x) = u_\varepsilon(\varepsilon^{1/p}x + x_\varepsilon)$. By a compactness argument, as in the proof of Proposition 4.2, we see that $U_\varepsilon(x) \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, where U with $0<U<\rho_2$ is a positive solution of the equation $\Delta_p U + f(U) = 0$ in $\mathbb{R}^N$. Moreover, it follows from $\int_\Omega \underline{u}_\varepsilon^p\,dx \leqslant C\varepsilon^{N/p}$ and the change of variable that $\int_{\mathbb{R}^N} U^p\,dx \leqslant C$. We know from the proof of (iii) of Theorem B that

$$\frac{\varepsilon}{p}\int_\Omega |D\underline{u}_\varepsilon|^p\,dx \leqslant C\varepsilon^{N/p}.$$

Thus, by the change of variable, we obtain $\int_{\mathbb{R}^N} |DU|^p\,dx \leqslant C$. Therefore, $U \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$.

Now we show that $f(U) \in L^1(\mathbb{R}^N)$. In fact, it follows from the assumptions in Theorem C that there exists a σ^* with $\rho_1+\sigma^* < \hat{\mu}$ such that $\Omega_{\varepsilon,\rho_1+\sigma^*}$ is a connected convex set. Then the continuity of $\underline{u}_\varepsilon$ implies that we can choose a convex domain $O_\varepsilon \subset\subset \Omega_{\varepsilon,\rho_1+\sigma^*}$ with boundary suitably smooth such that $\max_{\overline{\Omega_{\varepsilon,\rho_1+\sigma^*}\setminus O_\varepsilon}} \underline{u}_\varepsilon < \hat{\mu}$. We claim that

$$\underline{u}_\varepsilon \leqslant Ce^{-\sigma\varepsilon^{-1/p}} \quad \text{in} \quad \Omega\setminus\Omega_{\varepsilon,\rho_1+\sigma^*}, \tag{7.1}$$

where C and σ are independent of ε. In fact, for a fixed $x^* \in \partial O_\varepsilon$, without loss of generality, we may assume that $x^* = 0$ and $\nu(x^*) = e_1 = (1, 0, \ldots, 0)$, where $\nu(x)$ is the outward normal vector at $x \in \partial O_\varepsilon$. We choose $\gamma > 0$ such that

$$\Omega_1 := \{x \in \Omega,\ x \cdot e_1 = x_1 > 0\} \subset \{x \in \mathbb{R}^N : x \cdot e_1 = x_1,\ 0 < x_1 < \gamma\} \tag{7.2}$$

and $\partial\Omega \cap \{x_1 = \gamma\} \neq \varnothing$. Define

$$\Omega'_t = \Omega \cap \{x \in \mathbb{R}^N : x \cdot e_1 = x_1 > \gamma - t\}$$

and

$$w_{\theta, \varepsilon, t}(x) = w_{\theta, \varepsilon}(x_1 + t - \gamma) \qquad \text{for} \quad 0 \leqslant t \leqslant \gamma$$

where $w_{\theta, \varepsilon}$ is the solution of (6.6). Clearly $w_{\theta, \varepsilon, t}$ satisfies

$$\varepsilon \Delta_p u + g_\theta(u) = 0 \qquad \text{in} \quad \Omega'_t$$

and there exists a small $t_0 > 0$ (depending on θ and ε) such that $w_{\theta, \varepsilon, t} > \underline{u}_\varepsilon$ in Ω'_{t_0}. Arguments similar to those in the proof of Lemma 6.1 imply that

$$w_{\theta, \varepsilon, \gamma} \geqslant \underline{u}_\varepsilon \qquad \text{in} \quad \Omega'_\gamma. \tag{7.3}$$

Therefore, the compactness of ∂O_ε and the properties of $w_{\theta, \varepsilon, \gamma}$ imply that

$$\underline{u}_\varepsilon \leqslant \rho_0(\theta) + C e^{-\sigma \varepsilon^{-1/p}} \qquad \text{in} \quad \Omega \backslash \Omega_{\rho_1 + \sigma^*}, \tag{7.4}$$

where σ and C are independent of ε and θ. Letting $\theta \to 0$, we obtain our claim (7.1).

Now, with the help of (7.1), we show that

$$\int_\Omega \underline{u}_\varepsilon^{p-1}\, dx \leqslant C \varepsilon^{N/p} \tag{7.5}$$

for C independent of ε. In fact, from the proof of (iii) of Theorem B we have

$$|\Omega_{\varepsilon, \rho_1 + \sigma^*}| \leqslant C \varepsilon^{N/p}.$$

On the other hand, by (7.1) we also have

$$\int_\Omega \underline{u}_\varepsilon^{p-1}\, dx = \int_{\Omega \backslash \Omega_{\varepsilon, \rho_1 + \sigma^*}} \underline{u}_\varepsilon^{p-1}\, dx + \int_{\Omega_{\varepsilon, \rho_1 + \sigma^*}} \underline{u}_\varepsilon^{p-1}\, dx \leqslant C \varepsilon^{N/p}. \tag{7.6}$$

This implies that $\int_{\mathbb{R}^N} U^{p-1} = dx \leqslant C$. Since $|f(s)| \leqslant Ms^{p-1}$ for $s \in [0, \rho_2]$, we have

$$\int_{\mathbb{R}^N} |f(U)|\, dx \leqslant M \int_{\mathbb{R}^N} U^{p-1}\, dx \leqslant C. \tag{7.7}$$

Theorem D implies that U is radially symmetric (since the fact that $f'(s) < 0$ for s near 0 in (F_1) implies that there exists $\tilde{\delta} > 0$ such that f is nonincreasing for $0 < s < \tilde{\delta}$). Since $\max U \in (\rho_1, \rho_2 - \sigma)$, it follows from [20] that $U(x) \equiv w(r)$ is the unique positive (radial) solution of (1.2). ∎

We now give an estimate for $J_\varepsilon(\underline{u}_\varepsilon)$.

LEMMA 7.1. *We have*

$$0 < J_\varepsilon(\underline{u}_\varepsilon) \leqslant \varepsilon^{N/p}[I(U) + o(1)], \tag{7.8}$$

where $I(U) = \frac{1}{p} \int_{\mathbb{R}^N} |DU|^p - \int_{\mathbb{R}^N} F(U)$, $F(u) = \int_0^u f(s)\, ds$, *and* U, *with* $\max U \in (\rho_1, \rho_2 - \sigma)$, *is the unique positive (radial) solution of Eq.* (1.2) *in* $W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$.

Proof. The main idea in this lemma is similar to that in the proofs of Proposition 3.2 of [9] and Theorem 3.5 of [25]. Note that a_1, $\underline{u}_\varepsilon$ in Proposition 3.2 of [9] are 0 here. We have from [20] that the problem

$$\varepsilon \Delta_p u + f(u) = 0 \text{ in } B, \qquad u = 0 \quad \text{on } \partial B,$$

where B is the unit ball in $\mathbb{R}^N$, has exactly two positive radial solutions $\hat{u}_\varepsilon^1$ and $\hat{u}_\varepsilon^2$ for ε sufficiently small. Moreover, we also know from [20] that $\hat{u}_\varepsilon^1$ and $\hat{u}_\varepsilon^2$ satisfy $(\hat{u}_\varepsilon^1)'(r) < 0$, $(\hat{u}_\varepsilon^2)'(r) < 0$ for $r \in (0, 1]$; $\hat{u}_\varepsilon^1$ is the maximal positive solution (and is close to ρ_2) on B; $\hat{\mu} < \max \hat{u}_\varepsilon^2 < \rho_2$ and if we let $\tilde{U}_\varepsilon(r) = \hat{u}_\varepsilon^2(\varepsilon^{1/p} r)$, then $\tilde{U}_\varepsilon \to U$ in $C^1_{\text{loc}}(0, \infty)$ as $\varepsilon \to 0$. The arguments in the proofs of Theorems 5.2 and B imply that $\hat{u}_\varepsilon^2$ is a mountain pass solution. (To obtain the symmetry results, in [20], the author used the results in [3] and [4].) Moreover, the proofs of Lemma 6.1 and (iii) of Theorem B imply that for any $0 < \sigma^* < \hat{\mu} - \rho_1$, there exists $\tilde{C} = C(\sigma^*) > 0$ independent of ε such that the set $\{r \in [0, 1) : \hat{u}_\varepsilon^2(r) > \rho_1 + \sigma^*\} \subset [0, \varepsilon^{1/p}\tilde{C})$. The proof of (7.1) implies that $\hat{u}_\varepsilon^2(r) \leqslant Ce^{-\sigma\varepsilon^{-1/p}}$ for $\varepsilon^{1/p}\tilde{C} \leqslant r \leqslant 1$, where C and σ are independent of ε. Hence if we use the radial functional $\hat{J}_\varepsilon$ for the equation and use the minimax on all nondecreasing paths joining 0 and $\hat{U}_\varepsilon(x)$, where

$$\hat{U}_\varepsilon(x) = \begin{cases} \hat{u}_{\varepsilon_0}^1((\varepsilon_0/\varepsilon)^{1/p}\, x), & \text{if } \ x \in (\varepsilon/\varepsilon_0)^{1/p}\, B, \\ 0, & \text{if } \ x \in B \backslash (\varepsilon/\varepsilon_0)^{1/p}\, B, \end{cases}$$

for some small $\varepsilon_0 > 0$ with $\varepsilon < \varepsilon_0$, the energy $\hat{c}_\varepsilon = \hat{J}_\varepsilon(\hat{u}_\varepsilon^2) = \varepsilon^{N/p}(I(U) + o(1))$. (This can easily be obtained by the same argument as in the proof of Proposition 3.2 of [9]. Note that $\hat{U}_\varepsilon$ is a radial function, U decays exponentially as $r \to \infty$, and $\hat{u}_\varepsilon^2$ is exponentially small in any closed interval in $(0, 1]$.) Define $p_1^\varepsilon(t)$, $\tilde{p}(t) = (1 - l(x))\, p_1^\varepsilon(t)$ to be as in the proof of Proposition 3.2 of [9]. Then, from the proof of Theorem 5.2, we have

$$J_\varepsilon(\tilde{p}(1)) = \hat{J}_\varepsilon(\hat{U}_\varepsilon) < 0.$$

Furthermore, also from that proof, we have $\|\tilde{p}(1)\| > \eta$, where η is as in that proof. Therefore,

$$J_\varepsilon(\tilde{p}(t)) = \hat{J}_\varepsilon(p_1^\varepsilon(t)).$$

Hence,

$$J_\varepsilon(\underline{u}_\varepsilon) = c_\varepsilon \leqslant \varepsilon^{N/p}(I(U) + o(1)). \quad \blacksquare$$

The next lemma is similar to Lemma 4.2 of [30].

LEMMA 7.2. *Let $\phi \in C^1(\overline{B_b})$ (B_b is the ball with centre 0 and radius b) be a radial function and satisfy $\phi'(0) = 0$ and $(|\phi'(r)|^{p-2}\, \phi'(r))' < 0$ for $0 \leqslant r \leqslant b$. Then there exists $\delta_1 > 0$ such that if $\psi \in C^1(\overline{B_b})$ satisfies*

(i) *$D\psi(0) = 0$ and*
(ii) *$\|\operatorname{div}(|D\psi|^{p-2}\, D\psi - |D\phi|^{p-2}\, D\phi)\|_{L^\infty(\overline{B_b})} < \delta_1$,*

then $D\psi \neq 0$ for $x \neq 0$.

Proof. By replacing ψ_{x_j} and ϕ_{x_j} in the proof of Lemma 4.2 of [30] by $|D\psi|^{p-2}\, \psi_{x_j}$ and $|D\phi|^{p-2}\, \phi_{x_j}$, the proof of this lemma follows easily from the cited proof. $\blacksquare$

Applying this lemma and Theorem D, we can show that if P_ε^1, P_ε^2 are two local maximum points of $\underline{u}_\varepsilon$, then

$$\varepsilon^{-1/p}\, |P_\varepsilon^1 - P_\varepsilon^2| \to \infty \tag{7.9}$$

as $\varepsilon \to 0$.

We first claim that if P_ε is a local maximum point of $\underline{u}_\varepsilon$, then $\underline{u}_\varepsilon(P_\varepsilon) > \rho_1$. In fact, since P_ε is a local maximum point of $\underline{u}_\varepsilon$, there is a ball $B_\xi = B_\xi(P_\varepsilon) \subset\subset \Omega$ with $\xi > 0$ sufficiently small (which may depend on ε) such that $\partial \underline{u}_\varepsilon / \partial \nu \leqslant 0$ on $\partial B_\xi(P_\varepsilon)$. Then multiplying both sides of (5.1) by $\underline{u}_\varepsilon$

and integrating on B_ξ, we obtain $\int_{B_\xi} f(\underline{u}_\varepsilon)\, \underline{u}_\varepsilon\, dx > 0$. This implies that there exists $0 < \xi_0 \leqslant \xi$ such that $f(\underline{u}_\varepsilon) > 0$ on B_{ξ_0}. Thus, $\underline{u}_\varepsilon(P_\varepsilon) > \rho_1$. This also implies that P^1_ε and P^2_ε are bounded away from $\partial\Omega$.

Now defining $U^1_\varepsilon(y) = \underline{u}_\varepsilon(\varepsilon^{1/p} y + P^1_\varepsilon)$, we see that $U^1_\varepsilon \to U$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $\varepsilon \to 0$. We also know that for any $R > 0$,

$$\|\operatorname{div}(|DU^1_\varepsilon|^{p-2}\, DU^1_\varepsilon - |DU|^{p-2}\, DU)\|_{L^\infty(\overline{B_R})} \to 0 \qquad \text{as} \quad \varepsilon \to 0$$

since $f(U^1_\varepsilon) \to f(U)$ in $\overline{B_R}$ as $\varepsilon \to 0$. To obtain (7.9), as in the proof of Lemma 4.2 of [30], we only need prove that U^1_ε has exactly one local maximum point in B_R. This fact can be obtained by arguments similar to those in the paragraph immediately after the proof of Lemma 4.2 of [30] and using Lemma 7.2. (Note that $(|U'(r)|^{p-2}\, U'(r))' < 0$ for $r \in [0, b]$ and some $b > 0$ (since $(|U'|^{p-2}\, U')'\,(0) < 0$ (see [20])), so that U has exactly one local maximum point in $\mathbb{R}^N$.)

The proof that $\underline{u}_\varepsilon$ has only one local maximum point in Ω (and thus the proof of Theorem C) is obtained by arguments similar to those in the proof of Theorem 1.1 of [9]. (Note that a_1, a_2, and $\underline{u}_\varepsilon$ in the cited theorem are 0, ρ_1, and 0, respectively, in our case.)

APPENDIX A

THEOREM A.1. *Let $p > 2$, Ω be a convex domain with smooth boundary. Let $u_{\varepsilon,\rho} \in C^1(\bar{\Omega})$ be a positive solution of the problem*

$$-\varepsilon \operatorname{div}((\rho + |Du|^{p-2})\, Du) = f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{A.1}$$

where $0 < \varepsilon < \varepsilon_0$ (with ε_0 as in the proof of Theorem B*), $\rho \in (0, 1/2)$. Then there exists $\sigma > 0$, independent of ρ and ε, such that for any $y \in \Omega^\sigma$, there is a fixed size cone $K_y \subset \Omega$ with vertex at y and $u_{\varepsilon,\rho}(y) = \min_{x \in K_y} u_{\varepsilon,\rho}(x)$.*

Theorem A.1 follows from Theorem A.2 and an idea similar to that in the proof of Lemma A in the Appendix of [25].

We first show that the moving plane method can be used near $\partial\Omega$. We first introduce some notation as in [14]. Let γ be a unit vector in $\mathbb{R}^N$ and T_λ be the hyperplane $\{\gamma \cdot x = \lambda\}$. For λ large, T_λ is disjoint from $\bar{\Omega}$. Let the plane move continuously toward Ω, preserving the same normal, that is, decrease λ, until T_λ begins to intersect $\bar{\Omega}$. From that moment, at every stage the plane T_λ cuts off from Ω an open cap $\Sigma(\lambda)$. Let $\Sigma'(\lambda)$ denote the reflection of $\Sigma(\lambda)$ in the plane T_λ. For convenience, let γ be the unit vector $(1, 0, \ldots, 0)$ and assume $\max_{x \in \bar{\Omega}} x_1 = \lambda_0$.

THEOREM A.2. *Let $p > 2$, Ω be a convex domain with smooth boundary $\partial\Omega$. Let $u_{\varepsilon,\rho} \in C^1(\bar{\Omega})$ be a positive solution of* (A.1). *Then there exist $\delta > 0$ independent of ρ and ε such that for $\lambda_0 - \delta \leqslant \lambda < \lambda_0$,*

$$\frac{\partial u_{\varepsilon,\rho}}{\partial x_1} < 0 \qquad \text{on } \Sigma(\lambda). \tag{A.2}$$

The lemma below shows that the moving plane procedure can be started.

LEMMA A.3. *Let $\varepsilon > 0$, $\rho \in (0, 1/2)$ be fixed, and $x_0 \in \partial\Omega$ with $\nu_1(x_0) > 0$. For some $\alpha > 0$ assume $u_{\varepsilon,\rho}$ is a C^1 function in $\overline{\Omega_\alpha}$ where $\Omega_\alpha = \Omega \cap \{|x - x_0| < \alpha\}$, $u_{\varepsilon,\rho} > 0$ in Ω, and $u_{\varepsilon,\rho} = 0$ on $\partial\Omega \cap \{|x - x_0| < \alpha\}$. Then there exists $\delta > 0$ (depending on ε and ρ) such that in $\Omega \cap \{|x - x_0| < \delta\}$, $(u_{\varepsilon,\rho})_{x_1} < 0$.*

Proof. Since $u_{\varepsilon,\rho} > 0$ in Ω, necessarily, $(u_{\varepsilon,\rho})_\nu \leqslant 0$ on $S := \partial\Omega \cap \{|x - x_0| < \alpha\}$, and hence $(u_{\varepsilon,\rho})_1 \equiv (u_{\varepsilon,\rho})_{x_1} \leqslant 0$ on S, for by decreasing α if necessary, we may assume $\nu_1 > 0$.

If the lemma were false there would be a sequence of points $x^j \to x_0$, with $(u_{\varepsilon,\rho})_1 (x^j) \geqslant 0$. For j large, the interval in the x_1 direction going from x^j to $\partial\Omega$ hits S at a point where $(u_{\varepsilon,\rho})_1 \leqslant 0$. Consequently, we have $(u_{\varepsilon,\rho})_1 (x_0) = 0$. Letting $g(s) = f(s) + Ms^{p-1}$ with $M > 0$ sufficiently large, from the assumptions of f, g is strictly increasing on $(0, \rho_2)$. Therefore,

$$-\varepsilon \operatorname{div}((\rho + |Du_{\varepsilon,\rho}|^{p-2})\, Du_{\varepsilon,\rho}) + Mu_{\varepsilon,\rho}^{p-1} \geqslant 0 \qquad \text{in } \Omega.$$

By a modified version of the strong maximum principle in [34] and [44] we get $(u_{\varepsilon,\rho})_\nu < 0$ on $\partial\Omega$ and so $(u_{\varepsilon,\rho})_1 (x_0) < 0$, a contradiction. ▮

The next lemma implies that the moving plane procedure can be continued.

LEMMA A4. *If for some λ satisfying $\lambda^* < \lambda < \lambda_0$,*

$$(u_{\varepsilon,\rho})_1 \leqslant 0 \text{ and } u_{\varepsilon,\rho}(x) \leqslant u_{\varepsilon,\rho}(x^\lambda) \text{ but } u_{\varepsilon,\rho}(x) \not\equiv u_{\varepsilon,\rho}(x^\lambda) \text{ in } \Sigma(\lambda),$$

then $u_{\varepsilon,\rho}(x) < u_{\varepsilon,\rho}(x^\lambda)$ in $\Sigma(\lambda)$ and $(u_{\varepsilon,\rho})_1 < 0$ on $\Omega \cap T_\lambda$.

Proof. Let $v_{\varepsilon,\rho}(x) = u_{\varepsilon,\rho}(x^\lambda)$ and $w_{\varepsilon,\rho}(x) = v_{\varepsilon,\rho}(x) - u_{\varepsilon,\rho}(x)$. Then $w_{\varepsilon,\rho}(x) \geqslant 0$ in $\Sigma(\lambda)$. We also have that $w_{\varepsilon,\rho}$ satisfies the equation

$$-\varepsilon\, \Sigma_{ij} \frac{\partial}{\partial x_i}\left[(\rho\delta_{ij} + a^{ij}_{\varepsilon,\rho}(x)) \frac{\partial w_{\varepsilon,\rho}}{\partial x_j} \right] + M(p-1)\, \xi^{p-2}_{\varepsilon,\rho}(x)\, w_{\varepsilon,\rho} \geqslant 0 \qquad \text{in } \Sigma(\lambda), \tag{A.3}$$

where $\xi_{\varepsilon,\rho}(x) \in (u_{\varepsilon,\rho}(x), v_{\varepsilon,\rho}(x))$ and $a^{ij}_{\varepsilon,\rho}(x)$ is as in [23]. Since $\rho > 0$, we have that the operator in (A.3) is uniformly elliptic. Since $w_{\varepsilon,\rho} = 0$ on $T_\lambda \cap \Omega$ it follows from the maximum principle that $w_{\varepsilon,\rho} > 0$ in $\Sigma(\lambda)$ and $(w_{\varepsilon,\rho})_1 > 0$ on T_λ. But on T_λ, $(w_{\varepsilon,\rho})_1 = -2(u_{\varepsilon,\rho})_1$, and the lemma is proved. ▮

Now, by Lemmas A.3 and A.4, we give the proof of Theorem A.2 using an idea similar to that of [14]. In fact, if

$$\lambda^{**} = \inf\{\lambda\colon \lambda < \lambda_0;\ (u_{\varepsilon,\rho})_1 < 0,\ u_{\varepsilon,\rho}(x) < u_{\varepsilon,\rho}(x^\lambda) \text{ for } x \in \Sigma(\lambda)\},$$

it follows from the arguments of [14] that at least one of the following occurs:

(i) $\Sigma'(\lambda^{**})$ becomes internally tangent to $\partial\Omega$ at some point P not on $T_{\lambda^{**}}$,

(ii) $T_{\lambda^{**}}$ is orthogonal to $\partial\Omega$ at some point $Q \in T_{\lambda^{**}} \cap \partial\Omega$.

Note that λ^{**} is independent of ε and ρ. The proof of Theorem A.2 now follows from the compactness of $\bar{\Omega}$.

In the proof of the following theorem we use C and σ to denote positive constants which are independent of ε but may change line to line.

THEOREM A.5. *Assume that* $p > 2$ *and* $h \in C^1((0,\infty)) \cap C^0([0,\infty))$ *satisfies*

(H$_1$) $h(0) > 0$, $\underline{\lim}_{s\to 0^+} h(s)/s^{p-1} > 0$.

(H$_2$) *There are exactly three numbers* $0 < z_1 < z_2 < z_3$ *such that* $h(z_i) = 0$ *for* $i = 1, 2, 3$, *there exists* $\hat{\sigma} > 0$ *and* $M > 0$ *sufficiently large such that* $h'(s) < 0$ *for* $s \in (z_3 - \hat{\sigma}, z_3)$ *and* $h(s) \leqslant M(z_3 - s)^{p-1}$ *for* $s \in [0, z_3]$.

(H$_3$) *There exists* $\tilde{\sigma} > 0$ *such that* $h'(s) < 0$ *for* $s \in (z_1 - \tilde{\sigma}, z_1 + \tilde{\sigma}) \setminus \{z_1\}$, $\lim_{s\to z_1^+} h(s)/(s - z_1)^{p-1} = -m_1 < 0$, *and* $h(s) \leqslant M(z_1 - s)^{p-1}$ *for* $s \in [0, z_1]$; $h(s) \leqslant M(z_2 - s)^{p-1}$ *for* $s \in [0, z_2]$.

(H$_4$) $\int_{z_1}^{z_3} h(s)\, ds > 0$.

Then for a given $\gamma > 0$, *there exists* $\bar{\varepsilon} > 0$ *such that for* $0 < \varepsilon < \bar{\varepsilon}$, *the ordinary differential equation*

$$\varepsilon(|w'|^{p-2} w')' + h(w) = 0 \quad \text{in } (0,\gamma), \qquad w'(0) = 0,\ w(\gamma) = z_1 \tag{A.4}$$

possesses a positive solution $w_\varepsilon(x)$ *with the following properties:*

(i) $w_\varepsilon(0) \to \mu$ *as* $\varepsilon \to 0$, $w_\varepsilon(0) \in (\mu, z_3)$, $w'_\varepsilon(x) < 0$ *for* $x \in (0,\gamma)$, *where* $\mu \in (z_2, z_3)$ *is the unique point such that* $\int_{z_1}^{\mu} h(s)\, ds = 0$.

(ii) $|w_\varepsilon - z_1| < Ce^{-\sigma/\varepsilon^{1/p}}$ *in any closed interval in* $(0, \gamma]$, *where* $C > 0$, $\sigma > 0$ *are independent of* ε. *Moreover,*

$$|w'_\varepsilon(\gamma)| \leqslant (C/\varepsilon)\, e^{-\sigma\varepsilon^{-1/p}}. \tag{A.5}$$

Proof. Let $\tilde{h}(s) = h(s + z_1)$. It follows from the assumptions on h that $z_2 - z_1$, $z_3 - z_1$ are the only two positive zeros of $\tilde{h}$. Moreover, $\int_0^{\mu - z_1} \tilde{h}(s)\, ds = 0$.

Now we consider the initial value problem

$$(|v'|^{p-2} v')' + \tilde{h}(v) = 0 \qquad \text{in } (0, R), \quad v(0) = \alpha, \quad v'(0) = 0. \tag{A.6}$$

Since $\tilde{h}$ satisfies the conditions (F_1)–(F_3) (with $\rho_1 = z_2 - z_1$, $\rho_2 = z_3 - z_1$), it is known from [20, 21] that there exists a unique solution v_α of (A.6) for $\alpha \in [\mu - z_1, z_3 - z_1)$.

For $\alpha \in [\mu - z_1, z_3 - z_1)$ define

$$\begin{aligned}
I_- &= \{\alpha\colon \exists\, 0 < R < \infty \text{ such that } v_\alpha(R) = 0\},\\
I_0 &= \{\alpha\colon v_\alpha(x) > 0,\ v'_\alpha(x) < 0 \text{ for every } x > 0 \text{ and } \lim_{x\to\infty} v_\alpha(x) = 0\},\\
I_+ &= \{\alpha\colon \exists\, R > 0 \text{ such that } v'_\alpha(R) = 0 \text{ and } 0 < v_\alpha(R) < \alpha\}.
\end{aligned}$$

From the first integral of the equation in (A.6), we see that

$$I_+ \cup I_- \cup I_0 = [\mu - z_1, z_3 - z_1), \qquad I_- = (\mu - z_1, z_3 - z_1), \qquad I_0 = \{\mu - z_1\}.$$

This implies that $v_{\mu - z_1}$ satisfies $\lim_{x\to\infty} v_{\mu - z_1}(x) = 0$. By arguments similar to [20], we see that

$$\limsup_{x\to\infty} v_{\mu - z_1}(x)\, e^{\left(\frac{m_1}{p-1} - \eta\right)^{1/p} x} < \infty \tag{A.7}$$

and

$$\lim_{x\to\infty} \frac{v'_{\mu - z_1}(x)}{v_{\mu - z_1}(x)} = -(m_1/(p-1))^{1/p} \tag{A.8}$$

for any $\eta \in (0, m_1/(p-1))$. This implies that $v_{\mu - z_1}$ decays exponentially fast at ∞.

Setting $y_\alpha(x) = z_1 + v_\alpha(x)$, y_α is a solution of

$$(|y'|^{p-2} y')' + h(y) = 0 \qquad y(0) = z_1 + \alpha, \qquad y'(0) = 0. \tag{A.9}$$

Moreover, $y'_{\mu - z_1}(x) < 0$ for $x \in (0, \infty)$ with $y_{\mu - z_1}(0) = \mu$ and $y_{\mu - z_1} - z_1$ decays exponentially fast at ∞. On the other hand, for $\alpha \in (\mu - z_1, z_3 - z_1)$, (A.9) has a solution y_α for which there exists R_α such that $y_\alpha(R_\alpha) = z_1$

and $y_\alpha > z_1$ in $[0, R_\alpha)$ (since $I_- = (\mu - z_1, z_3 - z_1)$). By the continuous dependence of y_α and R_α on α, we see that $\lim_{\alpha \to (\mu - z_1)^+} R_\alpha = \infty$ and $\lim_{\alpha \to (\mu - z_1)^+} y_\alpha(0) = \mu$. These imply that there exists $\bar{R}$ sufficiently large such that for any $R > \bar{R}$, (A.9) has a solution $y_R(x)$ with $y_R(R) = z_1$ and $y_R(0) \to \mu$ as $R \to \infty$. Defining $\varepsilon = (R/\gamma)^{-p}$, $\bar{\varepsilon} = (\bar{R}/\gamma)^{-p}$, and $w_\varepsilon(x) = y_R((R/\gamma)\, x)$ for $0 < \varepsilon < \bar{\varepsilon}$, we have that w_ε is a positive solution of (A.4) and $\lim_{\varepsilon \to 0} w_\varepsilon(0) = \mu$. This shows (i).

To prove (ii), we need the following sweeping out result.

Lemma A.6. *Assume that* $p > 2$ *and* h *satisfies* (H_1)–(H_4). *Let* $[a, b]$ *be a closed interval in* $\mathbb{R}$ *and* $\{v_t \in C^1(0, \gamma) \cap C^0([0, \gamma]),\ \max v_t < z_3 - z_1;\ t \in [a, b]\}$ *be a family of positive functions such that*

$$\varepsilon(|v_t'|^{p-2}\, v_t')' + h(z_1 + v_t) \geqslant 0$$

for all $t \in [a, b]$; v_t *satisfies* $v_t'(x) \leqslant 0$ *for* $r \in (0, \gamma)$. *If* $u \in C^1((0, \gamma)) \cap C^0([0, \gamma])$ *satisfies* $u > 0$ *in* $[0, \gamma)$ *and* $u'(x) < 0$ *for* $x \in (0, \gamma]$ *with* $\max u < z_3 - z_1$ *and*

$$\begin{aligned} &\varepsilon(|u'|^{p-2}\, u')' + h(z_1 + u) \leqslant 0 && \text{in } (0, \gamma) \\ &u \geqslant v_t && \text{at } x = \gamma \text{ for all } t \in [a, b] \text{ and} \\ &u \geqslant v_a && \text{in } [0, \gamma] \end{aligned}$$

then $u \geqslant v_b$ *in* $[0, \gamma]$.

Proof. The proof of this lemma is similar to the proof of the generalized sweeping principle of Serrin in [18]. Let $\tilde{h}(s) = h(z_1 + s)$. By the assumptions on h, there exists $M > 0$ such that $l(s) := \tilde{h}(s) + M s^{p-1}$ is strictly increasing on $[0, z_3 - z_1]$. Since $u'(x) < 0$, $v_t'(x) \leqslant 0$ for $x \in (0, \gamma]$, if $t \in [a, b]$ is such that $u \geqslant v_t$ in $[0, \gamma]$, then we have

$$-\varepsilon(|u'|^{p-2}\, u')' + M u^{p-1} \geqslant l(u) \geqslant l(v_t) \geqslant -\varepsilon(|v_t'|^{p-2}\, v_t')' + M v_t^{p-1}.$$

For any $x_0 > 0$ small, we have that

$$-\varepsilon(\Psi(\xi, u', v_t')\,(u - v_t)')' + M\Psi(\xi, u, v_t)\,(u - v_t) \geqslant 0 \qquad \text{in } [x_0, \gamma), \quad \text{(A.10)}$$

where $\Psi(\xi, y, z) = (p-1) \int_0^1 |\xi y + (1-\xi)\, z|^{p-2}\, d\xi$. The operator in (A.10) is uniformly elliptic. This gives the conclusion of this lemma in $[x_0, \gamma)$. Letting $x_0 \to 0$ completes the proof. ∎

Now we give the proof of (ii). We shall show that for any $x_1 \in (0, \gamma)$,

$$|w_\varepsilon - z_1| \leqslant C e^{-\left(\frac{m_1}{p-1} - \eta\right)^{1/p} \varepsilon^{-1/p}(x_1/2)}.$$

Define $\bar{w}_\varepsilon(x) = v_{\mu - z_1}(\varepsilon^{-1/p}x)$. Then $\bar{w}_\varepsilon(x) \leqslant Ce^{-(m_1/(p-1)-\eta)^{1/p}\varepsilon^{-1/p}x}$. On the other hand, $(w_\varepsilon - z_1)^{-1}(\mu - z_1) \to 0$ as $\varepsilon \to 0$. Thus, $w_\varepsilon(x_1/2) - z_1 < \mu - z_1$ for sufficiently small $\varepsilon > 0$, and consequently $w_\varepsilon(x) - z_1 < \bar{w}_\varepsilon(x - x_1/2)$ for $x \in (x_1/2, \gamma]$ by the sweeping out result in Lemma A.6. In fact, $v_t(x) := w_\varepsilon(x+t) - z_1$ for $t \in [0, \gamma]$ are a class of solutions of

$$\varepsilon(|v'|^{p-2}v')' + h(z_1 + v) = 0$$

and $v_t(\gamma) = 0$ for $t \in [0, \gamma]$ (we assume that $w_\varepsilon(x) - z_1 \equiv 0$ for $x > \gamma$).

Moreover, $v_\gamma = w_\varepsilon(x+\gamma) - z_1 < \bar{w}_\varepsilon(x - x_1/2)$ for $x \in (x_1/2, \gamma]$. Then, Lemma A.6 implies that $v_0(x) \leqslant \bar{w}_\varepsilon(x - x_1/2)$ and thus

$$w_\varepsilon(x_1) - z_1 \leqslant \bar{w}_\varepsilon(x_1/2) \leqslant Ce^{-\left(\frac{m_1}{p-1}-\eta\right)^{1/p}\varepsilon^{-1/p}(x_1/2)}.$$

This also implies that for any closed interval $K \subset (0, \gamma]$ there exists $\sigma > 0$ (depending on K) such that for $x \in K$,

$$|w_\varepsilon(x) - z_1| \leqslant Ce^{-\sigma\varepsilon^{-1/p}}.$$

This completes the proof of (ii).

Now we show (A.5). Define $w_\varepsilon(x) = z_1 + v_\varepsilon(x)$. Then v_ε satisfies the equation

$$\varepsilon(|v_\varepsilon'|^{p-2}v_\varepsilon')' + \tilde{h}(v_\varepsilon) = 0 \qquad \text{in } (0, \gamma). \tag{A.11}$$

Let $\psi_\varepsilon = -(|v_\varepsilon'/v_\varepsilon|^{p-2}v_\varepsilon'/v_\varepsilon)$. By a routine calculation, we have

$$\psi_\varepsilon^{-p/(p-1)}\psi_\varepsilon' = \frac{\tilde{h}(v_\varepsilon)}{\varepsilon\,|v_\varepsilon|^{p-1}}\psi_\varepsilon^{-p/(p-1)} + (p-1). \tag{A.12}$$

$$-(p-1)(\psi_\varepsilon^{-1/(p-1)})' = \frac{\tilde{h}(v_\varepsilon)}{\varepsilon\,|v_\varepsilon|^{p-1}}\psi_\varepsilon^{-p/(p-1)} + (p-1). \tag{A.13}$$

Since $\lim_{s\to 0}\tilde{h}(s)/s^{p-1} = -m_1 < 0$, $v_\varepsilon' < 0$ in $(0, \gamma]$, and $v_\varepsilon \to 0$ in $[x^*/2, x^*]$ as $\varepsilon \to 0$ for any $x^* \in (\gamma/2, \gamma)$, there exists $\bar{\varepsilon} > 0$ such that

$$\frac{\tilde{h}(v_\varepsilon)}{v_\varepsilon^{p-1}} > -(m_1 + 1) \qquad \text{in } [x^*/2, x^*]$$

for all $0 < \varepsilon < \bar{\varepsilon}$. Suppose that $\psi_\varepsilon^{-p/(p-1)} \leqslant (p-1)\,\varepsilon/(2(m_1+1))$ in $[x^*/2, x^*]$. Then

$$-(p-1)(\psi_\varepsilon^{-1/(p-1)})' = -\frac{(m_1+1)}{\varepsilon}\psi_\varepsilon^{-p/(p-1)} + (p-1), \tag{A.14}$$

and thus,

$$(\psi_\varepsilon^{-1/(p-1)})' < -\tfrac{1}{2} \qquad \text{in } [x^*/2, x^*]. \tag{A.15}$$

Integrating (A.15) on $(x^*/2, x^*)$, we have

$$\psi_\varepsilon^{-1/(p-1)}(x^*) - \psi_\varepsilon^{-1/(p-1)}(x^*/2) < -(x^*/4). \tag{A16}$$

This is a contradiction since under our assumption the left hand side of (A.16) tends to 0 as $\varepsilon \to 0$. This also implies that there exists $\tilde{x} \in (x^*/2, x^*)$ such that

$$\psi_\varepsilon^{-p/(p-1)}(\tilde{x}) > \frac{(p-1)\,\varepsilon}{2(m_1+1)}. \tag{A.17}$$

Then we have

$$|v_\varepsilon'(\tilde{x})|^p < \frac{2(m_1+1)}{(p-1)\,\varepsilon}\,(v_\varepsilon(\tilde{x}))^p < (C/\varepsilon)^p\, e^{-\frac{p\gamma}{4}\left(\frac{m_1}{p-1}-\eta\right)^{1/p}\varepsilon^{-1/p}} \tag{A.18}$$

since $\tilde{x} > x^*/2 > \gamma/4$. As $x^* \in (\gamma/2, \gamma)$ is arbitrary and $|v_\varepsilon'(\gamma)|^p < |v_\varepsilon'(x)|^p$ when x is near γ (note that $\tilde{h}(s) < 0$ for $s \in (0, z_2 - z_1)$) then (A.18) implies (A.5). This completes the proof of Theorem A.5. ∎

Choosing z_1 in Theorem A.5 to be 0, the following corollary is easily obtained.

COROLLARY A.7. *Assume $p > 2$ and f satisfies* (F_1)–(F_3). *Then for a given $\gamma > 0$, there exists $\bar{\varepsilon} > 0$ such that for $0 < \varepsilon < \bar{\varepsilon}$, the ordinary differential equation*

$$\varepsilon(|w'|^{p-2}\,w')' + f(w) = 0 \qquad \text{in } (0, \gamma), \quad w'(0) = 0, \quad w(\gamma) = 0 \tag{A.19}$$

possesses a positive solution $w_\varepsilon(x)$ with the following properties:

(i) *$w_\varepsilon(0) \to \hat{\mu}$ as $\varepsilon \to 0$, $w_\varepsilon(0) \in (\hat{\mu}, \rho_2)$, $w_\varepsilon'(x) < 0$ for $x \in (0, \gamma)$, where $\hat{\mu} \in (\rho_1, \rho_2)$ is the unique point such that $\int_0^{\hat{\mu}} f(s)\,ds = 0$.*

(ii) *$|w_\varepsilon| \leqslant C\,e^{-\sigma/\varepsilon^{1/p}}$ in any closed interval in $(0, \gamma]$. Moreover,*

$$|w_\varepsilon'(\gamma)| \leqslant (C/\varepsilon)\,e^{-\sigma\varepsilon^{-1/p}}, \tag{A.20}$$

where C and σ are independent of ε.

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